How much randomness is needed for high-confidence Monte Carlo integration?

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We study Monte Carlo methods for integrating smooth functions based on \( n \) function evaluations.

The classical way of assessing the precision \( \varepsilon \) of a randomized integration method is based on the root mean squared error (RMSE) or the mean error. Optimal Monte Carlo error rates in terms of the mean error are well known for classical Sobolev spaces \( W^r_p ([0,1]^d) \) and can be achieved with methods like control variates, in some cases also via stratified sampling. For spaces \( W^{r,\text{mix}}_p ([0,1]^d) \) of dominating mixed smoothness, optimal integration methods based on a randomly shifted and dilated Frolov rule have been found in [1, 2]. If, however, the error is measured in terms of small error \( \varepsilon \) with high probability \( 1 - \delta \), the so-called probabilistic error criterion, see [3], some of the aforementioned methods turn out to be suboptimal with the error \( \varepsilon = e(n, \delta) \) depending polynomially on \( \delta^{-1} \) instead of the polynomial dependence on \( \log \delta^{-1} \) we hope for. Optimality in classical Sobolev spaces can be restored for control variates employing the median-of-means, for stratified sampling concentration phenomena (Hoeffding’s inequality) can lead to optimality; in any event, the amount of random numbers in such optimal methods is proportional to \( n \). The randomized Frolov rule which uses only \( 2d \) random parameters independently of \( n \), however, turns out to be suboptimal with respect to the probabilistic error criterion.

This raises the question: How small can the probabilistic error be if we limit the amount of randomness? Restricted Monte Carlo methods that only use a small amount of random bits have been studied in [4] for the RMSE criterion. A similar study for the probabilistic error criterion of restricted Monte Carlo methods will be presented.

Joint work with: Daniel Rudolf.

References