Dual graphs of projective schemes

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Example: 27 lines

Let us take 27 lines $L_1, \ldots, L_{27}$ in $\mathbb{P}^3_{\mathbb{C}}$, and consider their union

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As a consequence of the results I will present today, in this case it must happen that **each line meets at least 10 of the others**...

On the other hand, as we know, on a smooth cubic $X \subseteq \mathbb{P}^3_{\mathbb{C}}$ there are exactly 27 lines, which can be read from the fact that X is the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ along 6 generic points.

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From the "blow-up interpretation", it is immediate to check that the line corresponding to the exceptional divisor of any of the 6 points meet *exactly* 10 of the others.

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The main motivation for this talk comes from the desire of understanding **how global properties of $X \subseteq \mathbb{P}^n$ influence the combinatorial configuration of its irreducible components.**

One way to make precise the concept of "combinatorial configuration of its irreducible components" is by meaning of the dual graph of X ....

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Given $X \subseteq \mathbb{P}^n$ and the unique saturated homogeneous ideal $I_X \subseteq S = K[x_0, \ldots, x_n]$ s.t. $X = \mathrm{Proj}(S/I_X)$, let us recall that $X \subseteq \mathbb{P}^n$ is **arithmetically Cohen-Macaulay** (resp. **arithmetically Gorenstein**) if S/I_X is Cohen–Macaulay (resp. Gorenstein).

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On the other hand ......

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Theorem A, Benedetti-Bolognese-V. 2015

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For any connected graph G, there exists a reduced aCM curve $C \subseteq \mathbb{P}^n$ such that $G(C) = G$. Furthermore, $\text{reg}(C) = \text{reg}(I_C) = 3$ and the irreducible components of C are rational normal curves no 3 of which meet at one point.

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For a connected graph G, the following are equivalent:

- There is a curve $C \subseteq \mathbb{P}^n$ such that no 3 of its irreducible components meet at one point, $\text{reg}(C) = 2$, and $G(C) = G$.

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- G is a tree.

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Connectivity of graphs

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Menger theorem (Max-flow-min-cut).

A graph is d-connected iff between any 2 vertices one can find d vertex-disjoint paths.

From schemes to graphs

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Let $X \subseteq \mathbb{P}^n$ be an arithmetically Gorenstein projective scheme such that $\mathrm{reg}(X) = \mathrm{reg}(I_X) = r + 1$. If $\mathrm{reg}(\mathfrak{q}) \leq \delta$ for all primary components $\mathfrak{q}$ of I_X, then $G(X)$ is $\lfloor (r + \delta - 1)/\delta \rfloor$-connected.

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Benedetti-V. 2014

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She ran lots of examples choosing $\ell_i = x_0 + ix_1 + i^2x_2 + i^3x_3$ (or other variations), but in all tested cases C aCM $\implies k \leq 2$.

Hirsch embeddings

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Be careful:

- There exist nonreduced complete intersections $C \subseteq \mathbb{P}^3$ such that $C_{\mathrm{red}} \subseteq \mathbb{P}^3$ is a line arrangement and $\operatorname{diam}(G(C))$ is arbitrarily large.
- For large n, there are arithmetically Gorenstein line arrangements in $\mathbb{P}^n$ that are not Hirsch (Santos).

Hirsch embeddings

Many projective embeddings, however, are Hirsch:

Adiprasito–Benedetti 2014

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Benedetti–V. 2014

If X is an arrangement of lines, no 3 of which meet in the same point, canonically embedded in $\mathbb{P}^n$, then $X \subseteq \mathbb{P}^n$ is Hirsch.

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Many projective embeddings, however, are Hirsch:

Adiprasito–Benedetti 2014

If $X \subseteq \mathbb{P}^n$ is aCM and I_X is a monomial ideal generated by quadrics, then $X \subseteq \mathbb{P}^n$ is Hirsch.

Benedetti–V. 2014

If X is an arrangement of lines, no 3 of which meet in the same point, canonically embedded in $\mathbb{P}^n$, then $X \subseteq \mathbb{P}^n$ is Hirsch.

Conjecture: Benedetti–V. 2014

If $X \subseteq \mathbb{P}^n$ is a (reduced) aCM scheme and I_X is generated by quadrics, then $X \subseteq \mathbb{P}^n$ is Hirsch.

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This implies that the question above would admit a positive answer in dimension 2 if the EG conjecture was true in dimension 2 in its full generality (not only for irreducible surfaces).

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