Realizing graphs of genus 4 as skeleta of tropical space curves

Simon Hampe, TU Berlin
(joint work in progress with Ralph Morrison)

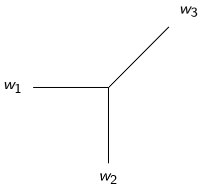
5. Jahreskonferenz des DFG Schwerpunktprogramms 1489

28. September 2015

What are tropical curves? - Answer I

Definition

A tropical curve C in $\mathbb{R}^n$ is a pure one-dimensional rational polyhedral complex with integer weights on its edges that make it *balanced* at each vertex.



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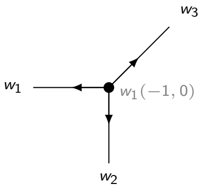

$w_1(-1,0) + w_2(0,-1) + w_3(1,1) \overset{!}{=} (0,0)$

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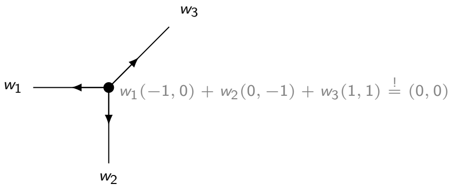


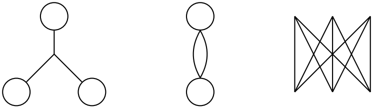
Definition

C is called *smooth*, if all weights are 1 and that's the only possible weight function up to multiples.

What are tropical curves? - Answer II

Definition

A tropical curve Γ is a finite abstract metric graph such that each vertex is at least trivalent.



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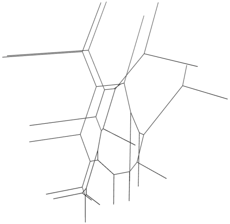

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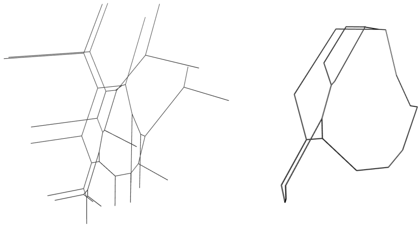

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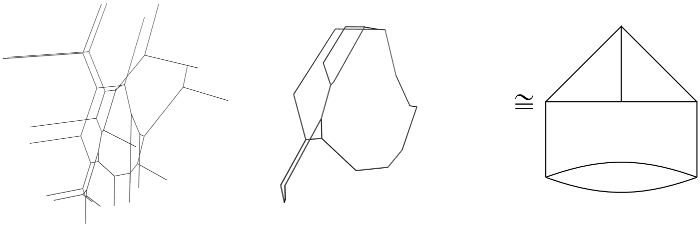

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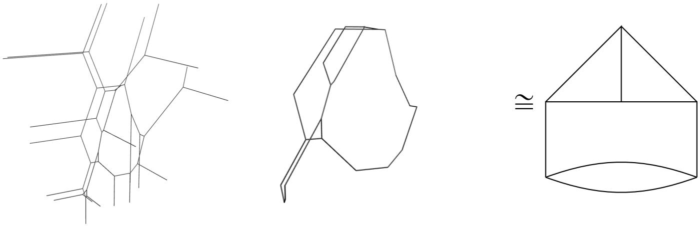

We call $sk(C)$ the *skeleton* of the curve C and say Γ is *realisable* if it lies in $\mathrm{Im}(sk)$.

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- What is the image of sk, i.e. what abstract curves (of fixed genus g) are realisable (by smooth curves in fixed $\mathbb{R}^n$)?
- What is the dimension of the set of realisable curves in the moduli space of abstract curves (of fixed genus g)?

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- They show that in terms of volume in the moduli space, the number of graphs realisable by planar curves decreases rapidly (e.g. for $g = 4$ only 0.5%) and eventually becomes lower-dimensional.

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- $\implies$ Pretend this is true in tropical geometry as well.

Tropical hypersurfaces

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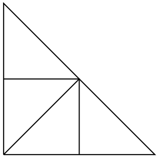
A smooth tropical hypersurface of degree d in $\mathbb{R}^n$ is the dual of a regular unimodular subdivision of $d\Delta_n$, the d-th dilate of the n-dimensional simplex.



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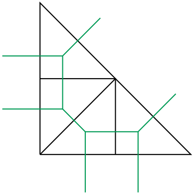

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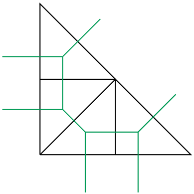

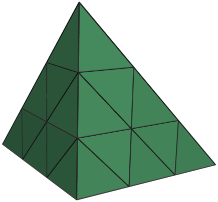

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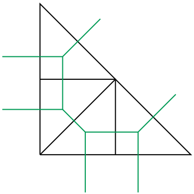

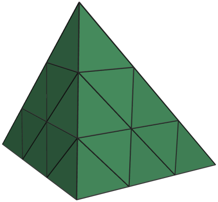

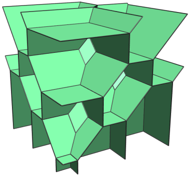

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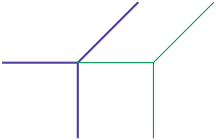

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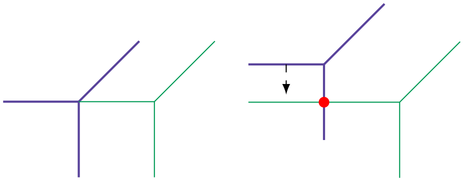

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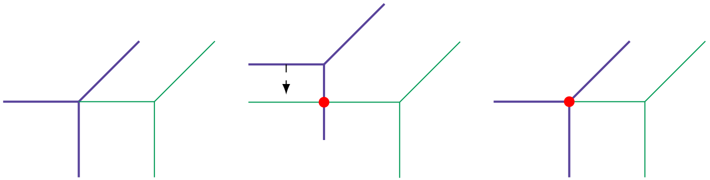

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- **The Cayley trick:** Any subdivision of $\mathcal{C}$ induces a *mixed subdivision* of $P + Q \cong \mathcal{C} \cap \{x_1 = x_2 = 1/2\}$.

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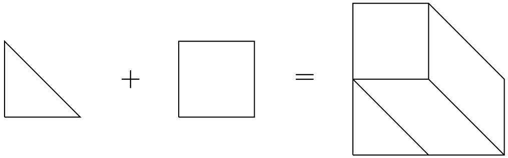

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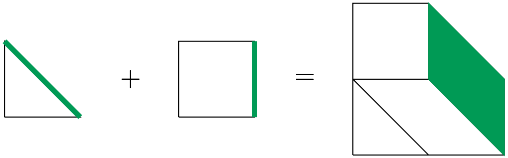

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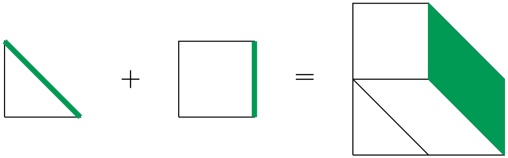

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Conclusion

We can find smooth genus-4 curves in $\mathbb{R}^3$ by finding regular unimodular triangulations of $\mathcal{C}(2\Delta_3, 3\Delta_3)$.

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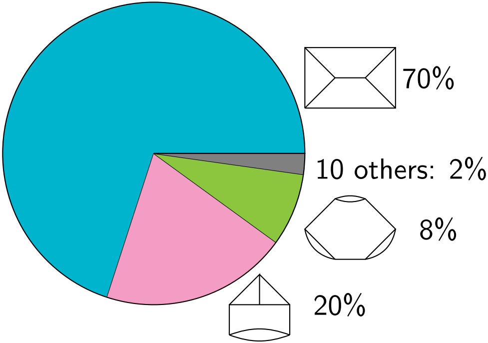
Result

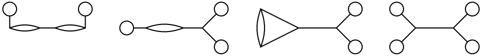
In our case, the success rate was $\sim 8\%$, which gave us about 1.5 million unimodular triangulations in 3 days.

Tropical pie

The result of the analysis of the 1.5 million triangulations:



We found 13 of the 17 possible types of trivalent genus 4 graphs.
The missing ones:



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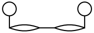
- The graph , while not occuring, is known to be realisable.

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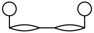
- The graph , while not occuring, is known to be realisable.
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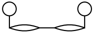
- The graph , while not occuring, is known to be realisable.
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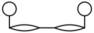
- The graph , while not occuring, is known to be realisable.
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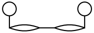
- The graph , while not occuring, is known to be realisable.
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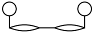
- The graph , while not occuring, is known to be realisable.
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Conclusion II

Need a better sample of triangulations of $3\Delta_3$.

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