COMPUTATIONAL ASPECTS OF RATIONAL CHEREDNIK ALGEBRAS

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COMPUTING IN
RATIONAL CHEREDNIK ALGEBRAS

Setup

K	a field contained in $\mathbb{C}$
$\mathfrak{h}$	a finite-dimensional K-vector space
W	a finite subgroup of $\mathrm{GL}(\mathfrak{h})$ generated by reflections
R	a commutative K-algebra ($R = K$ usually)
t	an element of R
c	a map $\mathrm{Ref}(W) \to R$, invariant under W-conjugation

Definition (Etingof–Ginzburg)

The rational Cherednik algebra $\mathsf{H}_{t,c}$ is the quotient of $R\langle \mathfrak{h} \oplus \mathfrak{h}^* \rangle \rtimes W$ by

$$[x, x'] = 0 = [y, y'] \quad \forall x, x' \in \mathfrak{h}^*,\ y, y' \in \mathfrak{h}\ ,$$

$$[y, x] = t\langle y, x \rangle - \sum_{s \in \mathrm{Ref}(W)} c(s) \frac{\langle y, \alpha_s \rangle \langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} s \quad \forall x \in \mathfrak{h}^*,\ y \in \mathfrak{h}\ ,$$

where $K\alpha_s^\vee = \mathrm{Im}(s - 1)$ and $K\alpha_s = \mathrm{Im}(s^* - 1)$.

Choose a basis $\mathbf{y} := (y_i)_{i=1}^n$ of $\mathfrak{h}$, and let $\mathbf{x} := (x_i)_{i=1}^n$ be the dual basis.

Due to the ordering of $\mathbf{y}$ we get an R-linear section

$$\begin{array}{ccc} R\langle \mathfrak{h} \oplus \mathfrak{h}^* \rangle \rtimes W & \twoheadrightarrow & R[\mathfrak{h} \oplus \mathfrak{h}^*] \rtimes W \simeq_R R[\mathfrak{h} \oplus \mathfrak{h}^*]W \\ & \overset{s_{\mathbf{y}}}{\dashleftarrow} & \\ \mathbf{x}^\alpha \mathbf{y}^\beta w & \longleftarrow\!\!\shortmid & \mathbf{x}^\alpha \mathbf{y}^\beta w \end{array}$$

Let $N_{\mathbf{y}}$ be the image of $s_{\mathbf{y}}$, a free R-module with basis $(\mathbf{x}^\alpha \mathbf{y}^\beta w)_{\alpha,\beta,w}$.

Theorem (Etingof–Ginzburg, Ram–Shepler, T.)

$\pi : R\langle \mathfrak{h} \oplus \mathfrak{h}^* \rangle \rtimes W \twoheadrightarrow \mathrm{H}_{t,c}$ induces $\xi_{\mathbf{y}} : N_{\mathbf{y}} \xrightarrow{\sim} \mathrm{H}_{t,c}$ as R-modules.

Challenge

Write a computer program that computes the basis representation of $\xi_{\mathbf{y}}^{-1}(\pi(a))$ for all a, or implement the deformed multiplication on $R[\mathfrak{h} \oplus \mathfrak{h}^*]W$.

One competitor

```
##############################################################
#  CHAMP (CHerednik Algebra Magma Package)                   #
#  Version v1.5-197-g10ce747                                 #
#  Copyright (C) 2013-2015 Ulrich Thiel                      #
#  See LMS J. Comput. Math. 18 (2015), no. 1, 266-307        #
#  http://thielul.github.io/CHAMP/                           #
##############################################################
> W  := ShephardTodd(2,1,2); // Weyl group B2
> H  := RationalCherednikAlgebra(W,0); // c generic, t=0
> eu := EulerElement(H); eu;
[1 0]                          [0 1]                 [1  0]
[0 1]*(y1*x1 + y2*x2) + [1 0]*(1/2*c1) + [0 -1]*(1/2*c2)
+
[-1  0]                    [0  -1]
[ 0  1]*(1/2)*(c2) + [-1  0]*(1/2*c1)
> IsCentral(eu);
true
```

```
> eu^2;
[ 0  -1]               [ 1   0]                    [ 0   1]
[ 1   0]*(c1*c2) + [ 0  -1]*(c2*y1*x1) + [-1   0]*(c1*c2)
+
[1  0]
[0  1]*(y1^2*x1^2 + 2*y1*y2*x1*x2 + y2^2*x2^2 + 1/2*c1^2 + 1/2*c2^2)
+
[ 0  -1]
[-1   0]*(1/2*c1*y1*x1 - 1/2*c1*y1*x2 - 1/2*c1*y2*x1 + 1/2*c1*y2*x2)
+
[-1   0]                             [-1  0]
[ 0  -1]*(1/2*c1^2 + 1/2*c2^2) + [0   1]*(c2*y2*x2)
+
[0  1]
[1  0]*(1/2*c1*y1*x1 + 1/2*c1*y1*x2 + 1/2*c1*y2*x1 + 1/2*c1*y2*x2)
```

COMPUTING THE CALOGERO–MOSER SPACE

We concentrate on $t = 0$ now.

Recall the relations of $\mathsf{H}_{0,c}$:

$$[x, x'] = 0 = [y, y'] \quad \forall x, x' \in \mathfrak{h}^*,\ y, y' \in \mathfrak{h}\ ,$$

$$[y, x] = -\sum_{s \in \mathrm{Ref}(W)} c(s) \frac{\langle y, \alpha_s \rangle \langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} s \quad \forall x \in \mathfrak{h}^*,\ y \in \mathfrak{h}\ .$$

Hence, $\mathsf{H}_{0,0} = R[\mathfrak{h} \oplus \mathfrak{h}^*] \rtimes W$. The center of this algebra is $R[\mathfrak{h} \oplus \mathfrak{h}^*]^W$.

So,

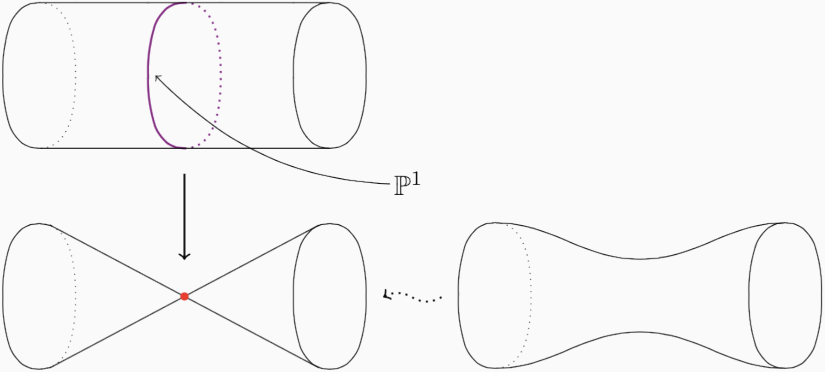
$$\mathrm{Spec}\, \mathsf{Z}(\mathsf{H}_{0,0}) = \mathrm{Spec}(R) \times (\mathfrak{h} \oplus \mathfrak{h}^*)/W\ .$$

For $R = K = \mathbb{C}$ this is a symplectic singularity (Beauville).

In general, $X_c := \mathrm{Spec}\, \mathsf{Z}(\mathsf{H}_{0,c})$ yields a flat family of deformations of $\mathrm{Spec}(R) \times (\mathfrak{h} \oplus \mathfrak{h}^*)/W$ over $\mathbb{A}_R^{\#\mathrm{Ref}(W)/W}$. Call X_c a Calogero–Moser space.

Example (Type A_1 Kleinian singularity)

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \subseteq \mathrm{SL}_2(\mathbb{C})$$



Challenge

Write a computer program to compute an explicit presentation of X_c.

First step

Determine generators of the center $\mathrm{Z}_{0,c}$ of $\mathrm{H}_{0,c}$.

Idea

1. Compute fundamental invariants of $\mathrm{Z}_{0,0} = K[\mathfrak{h} \oplus \mathfrak{h}^*]^W$.
2. Deform these to elements of $\mathrm{Z}_{0,c}$.

Algorithms for step 1 exist already (Singular, Magma, etc.).

Recall that $\mathsf{H}_{0,c} \simeq R[\mathfrak{h} \oplus \mathfrak{h}^*]W$ as R-modules. Hence, $h \in \mathsf{H}_{0,c}$ can be written as

$$h = \sum_{w \in W} h_w w$$

for some unique $h_w \in R[\mathfrak{h} \oplus \mathfrak{h}^*]$.

We thus have an R-linear map

$$\begin{array}{rcl} \mathrm{Trunc} : \mathsf{H}_{0,c} & \longrightarrow & R[\mathfrak{h} \oplus \mathfrak{h}^*] \\ h & \longmapsto & h_1 \ . \end{array}$$

Lemma (Bonnafé–T.)

Trunc restricts to an isomorphism $\mathsf{Z}_{0,c} \simeq R[\mathfrak{h} \oplus \mathfrak{h}^*]^W = \mathsf{Z}_{0,0}$.

We need to know how to compute the inverse map.

Consider the generic case:

$\mathbf{R} := K[\mathbf{c}_s \mid s \in \mathrm{Ref}(W)/W]$ the polynomial ring of rank $\#\mathrm{Ref}(W)/W$,
$\mathbf{c} : \mathrm{Ref}(W) \to \mathbf{R}$, $\mathbf{c}(s) := \mathbf{c}_s$,
$\mathbf{H} := \mathrm{H}_{0,\mathbf{c}}$,
$\mathbf{Z} := \mathrm{Z}_{0,\mathbf{c}}$.

Choose a regular vector $y_{reg} \in \mathfrak{h}^{reg}$, i.e., ${}^{w}y_{reg} \neq y_{reg}$ for all $w \in W \setminus \{1\}$.

If $z \in \mathbf{Z}$, then

$$z_w(y_{reg} - {}^{w}y_{reg}) = -\sum_{s \in \mathrm{Ref}(W)} [y_{reg}, z_{s^{-1}w}]_s \ .$$

Hence,

$$z_w^{\langle r\rangle}(y_{reg} - {}^w y_{reg}) = -\sum_{s\in \mathrm{Ref}(W)} [y_{reg}, z_{s^{-1}w}^{\langle r-1\rangle}]_s \ ,$$

where $z_w^{\langle r\rangle}$ is the part of $z_w \in \mathbf{R}[\mathfrak{h}\oplus\mathfrak{h}^*] \simeq K[(\mathbf{c}_s)_s \cup \mathbf{x} \cup \mathbf{y}]$ involving only $\mathbf{c}_s^l$ with $l \leq r$.

Can define an $(\mathbb{N}\times\mathbb{N})$-grading on $\mathbf{H}$ and $\mathbf{R}[\mathfrak{h}\oplus\mathfrak{h}^*]$ by putting $\mathfrak{h}^*$ in degree $(0,1)$, $\mathfrak{h}$ in degree $(1,0)$, W in degree $(0,0)$, and the $\mathbf{c}_s$ in degree $(1,1)$. The map $\mathrm{Trunc} : \mathbf{H} \to \mathbf{R}[\mathfrak{h}\oplus\mathfrak{h}^*]$ is graded.

Note: If z is bi-homogeneous of bi-degree (d,e), then $z_w^{\langle\delta\rangle} = z_w$ for all $w \in W$, where $\delta := \min(d,e)$.

Algorithm (Bonnafé–T.)

Let $f \in \mathbf{R}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ be bi-homogeneous of bi-degree (d, e). Let $z = \mathrm{Trunc}^{-1}(f)$ and write $z = \sum_{w \in W} z_w w$. Set $\delta := \min(d, e)$.

For each $0 \leq r \leq \delta$ we compute $z_w^{\langle r \rangle}$ for all $w \in W$ as follows:

1. $z_1^{\langle r \rangle} := f$ for all $0 \leq r \leq \delta$.
2. If $w \neq 1$ and $r = 0$, then $z_w^{\langle 0 \rangle} := 0$.
3. If $w \neq 1$ and $r > 0$, then

$$z_w^{\langle r \rangle} := -\frac{\sum_{s \in \mathrm{Ref}(W)} [y_{reg}, z_{s^{-1}w}^{\langle r-1 \rangle}]_s}{y_{reg} - {}^w y_{reg}} \; .$$

Then $z_w = z_w^{\langle \delta \rangle}$.

This algorithm is implemented in CHAMP:

```
> W  := ShephardTodd(2,1,2); // Weyl group B2
> H  := RationalCherednikAlgebra(W,0); // c generic, t=0
> CenterGenerators(~H); //output omitted
```

The algorithm is quite efficient. It works even for $W = F_4$!

We also have an algorithm to deform defining relations of $K[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ to defining relations of $Z_{0,c}$. (Details omitted...). In CHAMP:

```
> W  := ShephardTodd(2,1,2); // Weyl group B2
> H  := RationalCherednikAlgebra(W,0); // c generic, t=0
> CenterPresentation(H);
Ideal of Polynomial ring of rank 8 over Multivariate rational function field of rank 2 over Rational Field
Variables: Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8
Basis:
[
-Z1^2*Z3 + Z1*Z2^2 + Z1*Z6 + 1/2*c1^2*Z1 - 2*Z2*Z5 + Z3*Z4,
Z1*Z2*Z3 - Z1*Z7 - Z2^3 + 2*Z2*Z6 + c2^2*Z2 - Z3*Z5,
Z1*Z3^2 - Z1*Z8 - Z2^2*Z3 + 2*Z2*Z7 - Z3*Z6 - 1/2*c1^2*Z3,
-1/2*Z1^2*Z6 + (1/4*c1^2 - 1/4*c2^2)*Z1^2 + Z1*Z2*Z5 - Z1*Z3^3 + Z1*Z3*Z8 + Z2^2*Z3^2 - 1/2*Z2^2*Z4
- 2*Z2*Z3*Z7 + Z3^2*Z6 + 1/2*c1^2*Z3^2 + Z4*Z6 + 1/2*c2^2*Z4 - Z5^2,
Z1^2*Z2*Z3 - Z1^2*Z7 - Z1*Z2^3 + Z1*Z2*Z6 + (1/2*c1^2 + 1/2*c2^2)*Z1*Z2 + Z2^2*Z5 - Z2*Z3*Z4 + Z4*Z7
- Z5*Z6 - 1/2*c1^2*Z5,
-1/2*Z1^2*Z3^2 + 1/2*Z1^2*Z8 + 1/4*c2^2*Z1*Z3 + 1/2*Z2^4 - 1/2*Z2^2*Z6 + (-3/4*c1^2 - 1/2*c2^2)*Z2^2
- Z2*Z3*Z5 + Z3^2*Z4 - Z4*Z8 + Z5*Z7 + (1/2*c1^2 - 1/2*c2^2)*Z6 + 1/4*c1^4 - 1/4*c1^2*c2^2,
Z1^2*Z3^2 - Z1^2*Z8 - Z2^4 + 2*Z2^2*Z6 + (c1^2 + c2^2)*Z2^2 - Z3^2*Z4 + Z4*Z8 - Z6^2 - c1^2*Z6 - 1/4*c1^4,
-Z1*Z2*Z3^2 + Z1*Z2*Z8 + Z2^3*Z3 - Z2^2*Z7 - Z2*Z3*Z6 + (-1/2*c1^2 - 1/2*c2^2)*Z2*Z3 + Z3^2*Z5 - Z5*Z8 +
Z6*Z7 + 1/2*c1^2*Z7,
1/2*Z1*Z3^3 - 1/2*Z1*Z3*Z8 - 1/2*Z2^2*Z3^2 - 1/2*Z2^2*Z8 + 2*Z2*Z3*Z7 - Z3^2*Z6 - 1/4*c2^2*Z3^2 + Z6*Z8 -
Z7^2 + 1/2*c2^2*Z8
]
```

COMPUTING THE CALOGERO–MOSER FAMILIES

Recall: $\mathrm{H}_{0,c} \simeq R[\mathfrak{h} \oplus \mathfrak{h}^*] \rtimes W \simeq R[\mathfrak{h}] \otimes_R RW \otimes_R R[\mathfrak{h}^*]$

Theorem (Etingof–Ginzburg, Gordon)

The algebra $D(W) := R[\mathfrak{h}]^W \otimes_R R[\mathfrak{h}^*]^W$ is central in $\mathrm{H}_{0,c}$, and $\mathrm{H}_{0,c}$ is a finite free $D(W)$-module.

Let $D(W)_+$ be the ideal of $D(W)$ with zero constant term.

The restricted rational Cherednik algebra is $\overline{\mathrm{H}}_c := \mathrm{H}_{0,c}/D(W)_+\mathrm{H}_{0,c}$.

We have $\overline{\mathrm{H}}_c \simeq R[\mathfrak{h}]_W \otimes_R RW \otimes_R R[\mathfrak{h}^*]_W$, where $R[\mathfrak{h}]_W := R[\mathfrak{h}]/R[\mathfrak{h}]_+^W R[\mathfrak{h}]$ is the coinvariant algebra (extended to R).

Assume that R is a field, e.g., $R = K$, or $R = \mathrm{Frac}(\mathbf{R}) = K((\mathbf{c}_s)_s)$.

Recall: $\overline{\mathrm{H}}_c \simeq R[\mathfrak{h}]_W \otimes_R RW \otimes_R R[\mathfrak{h}^*]_W$

Theorem (Gordon)

For $\lambda \in \mathrm{Irr}\, W$ let

$$\Delta_c(\lambda) := \overline{\mathrm{H}}_c \otimes_{R[\mathfrak{h}^*]_W \rtimes W} \lambda \, .$$

This $\overline{\mathrm{H}}_c$-module has absolutely simple head $\mathrm{L}_c(\lambda)$, and $\lambda \mapsto \mathrm{L}_c(\lambda)$ yields $\mathrm{Irr}\, W \simeq \mathrm{Irr}\, \overline{\mathrm{H}}_c$.

Corollary

The block structure of $\overline{\mathrm{H}}_c$ yields a partition CM_c of $\mathrm{Irr}\, W$ into so-called Calogero–Moser c-families.

Theorem

For $R = K = \mathbb{C}$ there is a canonical bijection $\mathrm{CM}_c \simeq X_c^{\mathbb{C}^*}$.

Challenge

Compute the Calogero–Moser c-families for all K-valued c.

Elementary fact

- Every $z \in \mathrm{Z}(\overline{\mathrm{H}}_c)$ acts by a scalar $\Omega_c^\lambda(z) \in R$ on $\Delta_c(\lambda)$ and on $\mathrm{L}_c(\lambda)$.
- $\lambda, \mu \in \mathrm{Irr}\, W$ lie in the same CM_c-family if and only if $\Omega_c^\lambda = \Omega_c^\mu$.

Theorem (general result by B. Müller)

$\lambda, \mu \in \mathrm{Irr}\, W$ lie in the same CM_c-family if and only if $\Omega_c^\lambda(z_i) = \Omega_c^\mu(z_i)$ for algebra generators $z_1, \ldots, z_r$ of $\mathrm{Z}(\mathrm{H}_{0,c})$.

As we can compute generators of $\mathrm{Z}(\mathrm{H}_{0,c})$ now, we can compute CM_c!

Conjecture (Gordon–Martino)

For Coxeter groups W the Calogero–Moser c-families equal the Lusztig c-families coming from the Hecke algebra of W for all $\mathbb{R}$-valued c.

Theorem (Etingof–Ginzburg, Gordon, Martino, Bellamy)

True for $W = A_n, B_n, D_n, I_2(m)$ (a lot of combinatorics).

Theorem (T.)

True for $W = H_3$ (using some elementary tricks).

Theorem (Bonnafé–T.)

True for $W = F_4$ (using our algorithm).

Theorem (Etingof–Ginzburg)

X_c carries a Poisson bracket $\{\cdot,\cdot\}$ deforming the canonical one on $\mathrm{Spec}(R) \times (\mathfrak{h} \oplus \mathfrak{h}^*)/W$.

The bracket $\{\cdot,\cdot\}$ comes from the commutator in $\mathsf{H}_{t,c}$ for $t \neq 0$. CHAMP can compute this explicitly.

Theorem (Brown–Gordon)

X_c admits a stratification into symplectic leaves.

Theorem (Bellamy)

For $R = K = \mathbb{C}$ the zero-dimensional symplectic leaves of X_c are contained in $X_c^{\mathbb{C}^*}$.

Call a Calogero–Moser c-family cuspidal if the corresponding point of X_c is a zero-dimensional symplectic leaf.

Conjecture (Bellamy–T.)

For Coxeter groups W the cuspidal Calogero–Moser c-families equal the cuspidal Lusztig c-families coming from the Hecke algebra of W for all $\mathbb{R}$-valued c.

Theorem (Bellamy–T.)

True for $W = A_n, B_n, D_n, I_2(m)$.

Lemma (Bonnafé–T.)

The CM_c-family of $\lambda \in \operatorname{Irr} W$ is cuspidal if and only if $\Omega_c^\lambda(\{z, z'\}) = 0$ for all $z, z' \in Z_{0,c}$. We can thus compute Cuspidal families.

Theorem (Bonnafé–T.)

True for $W = H_3, F_4$.

- Simple modules and decomposition matrices of baby Verma modules of $\overline{\mathrm{H}}_c$.
  CHAMP contains a characteristic zero MeatAxe variant which is able to decompose baby Verma modules $\Delta_c(\lambda)$ for arbitrary c, even over Frac($\mathbf{R}$). (Based on the ModFinder algorithm).
- Symplectic leaves of X_c.
- Conjectures about (Poisson) cohomology and ζ-fixed points of X_c by Bonnafé–Rouquier.
- Conjectures about Calogero–Moser cells and Calogero–Moser cellular characters by Bonnafé–Rouquier.
- Lie algebra structure on the tangent space of X_c in cuspidal points.
- ...

Problem

How to deal with all c?

Let $\mathbf{c} : \mathrm{Ref}(W) \to \mathbf{R} \subset \mathrm{Frac}(\mathbf{R})$ be the generic parameter.

An arbitrary $c : \mathrm{Ref}(W) \to K$ is obtained by reducing c modulo an ideal $\mathfrak{m}_c$ of $\mathbf{R}$.

Because $\mathbf{R}$ is normal, we have $\Omega_{\mathbf{c}}^{\lambda}(z) \in \mathbf{R}$ for all $z \in Z_{0,\mathbf{c}}$.

Theorem (Bonnafé–Rouquier, T.; general result)

$\lambda, \mu \in \mathrm{Irr}\, W$ lie in the same CM_c-family if and only if $\Omega_{\mathbf{c}}^{\lambda} \equiv \Omega_{\mathbf{c}}^{\mu} \bmod \mathfrak{m}_c$.

So, we start with determining $\Omega_{\mathbf{c}}^{\lambda}$ for all λ to get $\mathrm{CM}_{\mathbf{c}}$, and if λ, μ lie in distinct $\mathrm{CM}_{\mathbf{c}}$-families, their families will merge on the subschemes defined by $\Omega_{\mathbf{c}}^{\lambda} - \Omega_{\mathbf{c}}^{\mu}$.

These subschemes are conjectured to be hyperplanes (BR, T.).