Nilpotent associative algebras and coclass theory

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Introduction
An associative algebra over a field $\mathbb{F}$ is a vectorspace over $\mathbb{F}$ equipped with an associative multiplication.

It is NOT assumed that an associative algebra contains an identity element.
Examples

(1) Matrix algebras: subalgebras of $M_n(\mathbb{F})$, for example the upper triangular matrices with 0’s on the diagonal.

(2) Group algebras: $\mathbb{F}G$ where $G$ is a finite group; these algebras always have an identity.

(3) Quaternion algebras: certain 4-dimensional algebras; these also have an identity.
Nilpotency

An associative algebra $A$ is *nilpotent* if there exists $c \in \mathbb{N}$ so that every product of $c + 1$ elements in $A$ is zero. The smallest $c$ with this property is the *class* $cl(A)$ of $A$.

Power Ideals

For an algebra $A$ let $A^i$ be the ideal spanned by all products of $i$ elements in $A$. Then $A$ is nilpotent of class $c$ if and only if

$$A = A^1 > A^2 > \ldots > A^c > A^{c+1} = \{0\}.$$
Let $A$ be a finite dimensional nilpotent associative algebra. Then the *coclass* of $A$ is defined as

$$dim(A) - cl(A).$$
Example

Let $A$ be the subalgebra of $M_n(\mathbb{F})$ consisting of all upper triangular matrices with 0’s on the diagonal.

(1) $A$ is nilpotent of dimension $n(n - 1)/2$ and class $n - 1$.

(2) $A$ has coclass $n(n - 1)/2 - (n - 1) = (n - 1)(n - 2)/2$. 
Example

Let \( G \) be a finite \( p \)-group of order \( p^n \), \( \mathbb{F} \) a finite field of characteristic \( p \) and \( A = J(\mathbb{F}G) \).

(1) \( A \) is nilpotent of dimension \( p^n - 1 \) and class \( p^l - 1 \), where \( l \) is the length of the Jennings series of \( G \).

(2) \( A \) has coclass \( p^n - 1 - (p^l - 1) = p^n - p^l \).
Significance

Structure theory (Wedderburn/Jacobson)

Let $A$ be a finite dimensional associative algebra with identity.

1. $A/J(A)$ is a direct sum of full matrix algebras over skewfields.
2. $J(A)$ is a nilpotent associative algebra.
Classification

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General Aims

A wide open problem

Classify the finite dimensional nilpotent associative algebras over a field $\mathbb{F}$ up to isomorphism.
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Using the dimension

Dimension

Classify the nilpotent associative algebras over $\mathbb{F}$ of dimension $d$:

- $d = 1$ is trivial:
  there is just one such algebra $C_1 = \langle a \mid a^2 = 0 \rangle$.

- $d = 2$ is easy:
  there are two such algebras $C_1 \oplus C_1$ and $C_2 = \langle a \mid a^3 = 0 \rangle$.

- $d = 3$ is known (Willem de Graaf), but not easy:
  If $\mathbb{F}$ is infinite, then there are infinitely many algebras.
  If $\mathbb{F}$ is finite, then there are $|\mathbb{F}| + 6$ or $|\mathbb{F}| + 5$ algebras.

Proceed with this? — Seems daunting.

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Open Problem

Higman

The number of isomorphism types of algebras of dimension $n$ over $\mathbb{F}_q$ is PORC (Polynomial on residue classes).

Open Problem

Is the number of isomorphism types of nilpotent associative algebras of dimension $n$ over $\mathbb{F}_q$ a PORC function?
Using the coclass

**Question**

Is it possible to classify the finite dimensional nilpotent associative algebras over $\mathbb{F}$ of coclass $r$?

**Example: Coclass $r = 0$**

This is easy! The resulting algebras are $C_n := \langle a \mid a^{n+1} = 0 \rangle$ for $n \in \mathbb{N}_0$.

Sounds promising?
Why coclass?

Nilpotent Groups
Coclass theory has first been considered for finite $p$-groups, initiated by Leedham-Green and Newman. Result is a very useful structure theory!

Nilpotent Lie Algebras
It has also been considered for nilpotent Lie algebras, mainly due to Shalev and Zelmanov.

Nilpotent Associative Algebras
It seems a promising approach for associative algebras. The main aim of this DFG project was to investigate this.
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Coclass Graph

The coclass graph

Let $\mathbb{F}$ be a field and $r \in \mathbb{N}_0$. The graph $\mathcal{G}_{\mathbb{F}}(r)$ is defined by:

- Vertices correspond one-to-one to isomorphism types of finite dimensional nilpotent associative $\mathbb{F}$-algebras of coclass $r$;
- There is an edge $A \to B$ if $A \cong B/B^{cl}(B)$; that is, if $B$ is a descendant of $A$. 

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Examples

Small Coclass

- $G_F(0)$ is easy for all fields $F$.
- $G_F(1)$ is a tree with root $C_1 \oplus C_1$.
- $G_F(2)$ is again more complicated...
Observations

Observation

In these small examples there are always finitely many infinite paths starting from the root.

First step

Investigate the infinite paths in $\mathcal{G}_F(r)$!
Infinite paths

Pro-nilpotent algebras

Let $A_1 \to A_2 \to \ldots$ be an infinite path in $\mathcal{G}_F(r)$ and let $A$ be the inverse limit of this path. Then

(a) $A$ is an infinite dimensional associative algebra.

(b) $A/A^{i+\text{cl}(A)} \cong A_i$ for all large enough $i$; say $cc(A) = r$.

(c) Equivalent paths define isomorphic inverse limits.

Correspondence

The maximal infinite paths in $\mathcal{G}_F(r)$ correspond one-to-one to the isomorphism types of infinite dimensional pro-nilpotent associative $F$-algebras of coclass $r$. 
Some definitions

Formal power series

(a) Let \( \mathbb{F}[[t]] \) be the ring of formal power series over \( \mathbb{F} \).

(b) Let \( \mathbb{F}_o[[t]] \) be the ideal generated by \( t \) in \( \mathbb{F}[[t]] \).

Annihilators

For an algebra \( A \) is

(a) \( Ann(A) = \{a \in A \mid ab = ba = 0 \text{ for all } b \in A\} \).

(b) \( Ann_0(A) = \{0\} \) and \( Ann_i(A) = Ann(A/Ann_{i-1}(A)) \) for \( i \geq 1 \).

(c) \( Ann_*(A) = \bigcup_{i \in \mathbb{N}} Ann_i(A) \).
A structure theorem

The following theorem exhibits the structure of the inverse limits of the infinite paths in $G_{\mathbb{F}}(r)$.

**Theorem (Eick & Moede)**

Let $\mathbb{F}$ be a field and $r \in \mathbb{N}_0$. 
$A$ is isomorphic to the inverse limit of an infinite path in $G_{\mathbb{F}}(r)$ if and only if $\text{dim}(\text{Ann}_*(A)) = r$ and $A = \text{Ann}_*(A) \rtimes \mathbb{F}_0[[t]]$.

**In other words**

Let $\mathbb{F}$ be a field and $r \in \mathbb{N}_0$.
Each infinite path in $G_{\mathbb{F}}(r)$ can be constructed as split extension of an $r$-dimensional nilpotent algebra with $\mathbb{F}_0[[t]]$. 
Let $n_F(r)$ denote the number of maximal infinite paths in $G_F(r)$.

(a) $n_F(0) = 1$ for all fields $F$.

(b) $n_F(1) = 1$ for all fields $F$.

(c) $n_F(2) = \infty$ if $F$ is infinite and $n_F(2) = |F| + 4$ if $F$ is finite.

Theorem (Eick & Moede)

$n_F(r)$ is finite if and only if $r \leq 1$ or $F$ is finite.
Algorithms
Descendants

An associative algebra $B$ is a descendant of $A$ if $A \cong B/B^{\text{cl}(A)+1}$.

Descendant tree

Given $A$ in $\mathcal{G}_F(r)$ we denote with $\mathcal{T}_A$ the full subtree of $\mathcal{G}_F(r)$ of descendants of $A$.

Maximal descendant tree

A descendant tree $\mathcal{T}_A$ is maximal if it is not properly contained in another descendant tree; that is, if $A = \{0\}$ or $A/A^{\text{cl}(A)}$ has coclass smaller than $r$. 
Algorithm I

**Immediate descendants**

Let $\mathbb{F}$ be a finite field. Developed an effective algorithm to determine up to isomorphism all immediate descendants of an algebra $A$ (in $\mathcal{G}_\mathbb{F}(r)$).

**Ingredients**

(a) Compute $\text{Aut}(A)$.

(b) Compute the multiplication $M$ and the nucleus $N$ of $A$. $(N$ is a subspace of the finite dimensional vectorspace $M.)$

(c) Compute the natural action of $\text{Aut}(A)$ on $M$.

(d) Compute orbits and stabilizers of all supplements to $N$ in $M$. 
Algorithm II

**Theorem (Eick & Moede)**

Let $\mathbb{F}$ be a finite field. Then $G_{\mathbb{F}}(r)$ consists of finitely many maximal descendant trees. The roots of these trees have dimension at most $2r$.

**Roots**

Let $\mathbb{F}$ be a finite field. Developed an effective algorithm to determine up to isomorphism the roots of $G_{\mathbb{F}}(r)$. 
Application

Algorithm I and II allow to investigate $\mathcal{G}_F(r)$.

(a) Compute the roots of the maximal descendant trees.

(b) Compute iteratively immediate descendants of these roots and their descendants.

(c) Yields large finite parts of the infinite graph $\mathcal{G}_F(r)$. 
Experiments

Determined finite parts of $G_{F}(r)$ for many finite fields and various $r$.

Result

Many detailed insights into the structure of $G_{F}(r)$.
Periodic patterns

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Coclass trees

A descendant tree $T_A$ in $G_{\mathbb{F}}(r)$ is a \textit{coclass tree} if it has a unique infinite path.

Maximal coclass trees

A coclass tree is maximal if it is not properly contained in another coclass tree.

Theorem (Eick & Moede)

Let $\mathbb{F}$ be a finite field and $r \in \mathbb{N}_0$. Then $G_{\mathbb{F}}(r)$ consists of finitely many maximal coclass trees and finitely many other vertices.
Periodicity

Let $\mathcal{T}$ be a maximal coclass tree with root $A$ and infinite path $A = A_1 \rightarrow A_2 \rightarrow \ldots$.

(a) $\mathcal{T}$ is *virtually periodic* with period $d$ and periodic root $A_l$ if $\mathcal{T}_{A_i}$ and $\mathcal{T}_{A_{i+d}}$ are graph isomorphic for each $i \geq l$.

(b) $\mathcal{T}$ has *depth* $k$ if every vertex in $\mathcal{T}$ has distance at most $k$ from the infinite path.
Conjectures

Conjecture (Eick & Moede)

Let $\mathbb{F}$ be a finite field and $r \in \mathbb{N}_0$. Then each maximal coclass tree $\mathcal{T}$ in $\mathcal{G}_{\mathbb{F}}(r)$ is virtually periodic and has finite depth.

Conjecture (Eick & Moede)

Let $\mathbb{F}$ be a finite field and $r \in \mathbb{N}_0$. The infinitely many algebras in $\mathcal{G}_{\mathbb{F}}(r)$ can be described by finitely many parametrised presentations.

Implications

If the conjectures hold, then the infinitely many nilpotent associative $\mathbb{F}$-algebras of coclass $r$ can be classified!
Let $\mathbb{F}$ be a finite field. Then $\mathcal{G}_{\mathbb{F}}(1)$ consists of a single coclass tree. This is periodic with period $|\mathbb{F}| - 1$ and depth 1.
Coclass 2

Conjecture (Eick & Moede)

Let $\mathbb{F}$ be a finite field of char $p > 2$ and size $q$. Then $G_{\mathbb{F}}(2)$ consists of $3q + 6$ maximal descendant trees. Of these, $2q + 2$ are finite and $q + 4$ are maximal coclass trees. The maximal coclass trees are all virtually periodic with

(a) There are $q + 1$ maximal coclass trees of depth 1 and period $q - 1$;

(b) There is 1 maximal coclass tree of depth 1 and period 1;

(c) There are 2 maximal coclass trees of depth 2 and period $p(q - 1)$;
**Conjecture (Eick & Moede)**

Let $\mathbb{F}$ be a finite field of char $p = 2$ and size $q$. Then $G_{\mathbb{F}}(2)$ consists of $3q + 5$ maximal descendant trees. Of these, $2q + 1$ trees are finite and $q + 4$ are maximal coclass trees. The maximal coclass trees are all virtually periodic with

(a) There are $q + 3$ maximal coclass trees of depth 1 and period $q − 1$;

(b) There is 1 maximal coclass tree of depth 2 and period $p(q − 1)$;