IMPROVED PARALLEL GAUSSIAN ELIMINATION FOR GRÖBNER BASIS COMPUTATIONS IN FINITE FIELDS

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October 01, 2015

University of Kaiserslautern

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- New data for the Gröbner basis can then be read off the reduced matrix: Restore corresponding rows as polynomials.

Specialize Linear Algebra for reduction steps in GB computations.

IDEA BY FAUGÈRE & LACHARTRE

Specialize Linear Algebra for reduction steps in GB computations.

$$\begin{matrix} 1 & 3 & 0 & 0 & 7 & 1 & 0 \\ 1 & 0 & 4 & 1 & 0 & 0 & 5 \\ 0 & 1 & 6 & 0 & 8 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 1 \end{matrix}$$

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Try to exploit underlying GB structure.

Specialize Linear Algebra for reduction steps in GB computations.

$$
\begin{array}{rl}
\text{S-pair} & \left\{ \begin{array}{ccccccc} 1 & 3 & 0 & 0 & 7 & 1 & 0 \\ 1 & 0 & 4 & 1 & 0 & 0 & 5 \end{array} \right. \\
\text{S-pair} & \left\{ \begin{array}{ccccccc} 0 & 1 & 6 & 0 & 8 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 7 & 0 \end{array} \right. \\
\text{reducer} & \leftarrow \begin{array}{ccccccc} 0 & 0 & 0 & 0 & 1 & 3 & 1 \end{array}
\end{array}
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\text{S-pair} & \left\{ \begin{array}{c} \\ \\ \end{array} \right. & \begin{array}{c} 0 \\ 0 \end{array} & \begin{array}{c} 1 \\ 1 \end{array} & \begin{array}{c} 6 \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \end{array} & \begin{array}{c} 8 \\ 0 \end{array} & \begin{array}{c} 0 \\ 7 \end{array} & \begin{array}{c} 1 \\ 0 \end{array} \\
\text{reducer} & \leftarrow & 0 & 0 & 0 & 0 & 1 & 3 & 1
\end{array}
$$

Try to exploit underlying GB structure.

Main idea

Do a static reordering before the Gaussian Elimination to achieve a better initial shape. Invert the reordering afterwards.

1st step: Sort pivot and non-pivot columns

$$\begin{matrix} 1 & 3 & 0 & 0 & 7 & 1 & 0 \\ 1 & 0 & 4 & 1 & 0 & 0 & 5 \\ 0 & 1 & 6 & 0 & 8 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 1 \end{matrix}$$

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Pivot column

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FAUGÈRE-LACHARTE IDEA

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0 & 0 & 0 & 0 & 1 & 3 & 1
\end{matrix}
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FAUGÈRE-LACHARTRE IDEA

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0 & 0 & 0 & 0 & 1 & 3 & 1
\end{matrix}
\longrightarrow
\begin{array}{ccc|cccc}
1 & 3 & 7 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 4 & 1 & 0 & 5 \\
0 & 1 & 8 & 6 & 0 & 0 & 9 \\
0 & 1 & 0 & 0 & 0 & 7 & 0 \\
0 & 0 & 1 & 0 & 0 & 3 & 1
\end{array}
$$

Pivot column Non-Pivot column

FAUGÈRE-LACHARTRE IDEA

2nd step: Sort pivot and non-pivot rows

$$\begin{array}{ccc|cccc}
1 & 3 & 7 & 0 & 0 & 1 & 0 \\
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Pivot row

2nd step: Sort pivot and non-pivot rows

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1 & 3 & 7 & 0 & 0 & 1 & 0 \\
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0 & 0 & 1 & 0 & 0 & 3 & 1
\end{array}\right)
\longrightarrow
\left(\begin{array}{ccc|cccc}
1 & 0 & 0 & 4 & 1 & 0 & 5 \\
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\hline
1 & 3 & 7 & 0 & 0 & 1 & 0 \\
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\end{array}\right)
$$

Pivot row

Non-Pivot row

3rd step: Reduce lower left part to zero

$$\left.\begin{array}{ccc|cccc} 1 & 0 & 0 & 4 & 1 & 0 & 5 \\ 0 & 1 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & 1 \\ \hline 1 & 3 & 7 & 0 & 0 & 1 & 0 \\ 0 & 1 & 8 & 6 & 0 & 0 & 9 \end{array}\right.$$

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4th step: Reduce lower right part

$$
\left(\begin{array}{ccc|cccc}
1 & 0 & 0 & 4 & 1 & 0 & 5 \\
0 & 1 & 0 & 0 & 0 & 7 & 0 \\
0 & 0 & 1 & 0 & 0 & 3 & 1 \\
\hline
0 & 0 & 0 & 7 & 10 & 3 & 10 \\
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\end{array}\right)
$$

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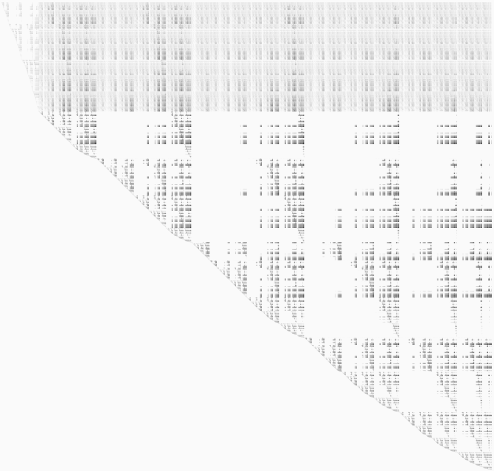
5th step: Remap columns and get new polynomials for GB out of lower right part.

SO, WHAT DO "REAL WORLD" MATRICES FROM GB COMPUTATIONS LOOK LIKE?

WHAT OUR MATRICES LOOK LIKE



Some data about the matrix:

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$$\text{full matrix:} \quad 21,182 \quad \times \quad 22,207$$

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- Size 55MB
- $24,006,869$ nonzero elements (density: 5%)
- Dimensions:

$$\begin{array}{rrcrl} \text{full matrix:} & 21,182 & \times & 22,207 & \\ \\ \text{upper-left:} & 17,915 & \times & 17,915 & \text{known pivots} \\ \text{lower-left:} & 3,267 & \times & 17,915 & \\ \text{upper-right:} & 17,915 & \times & 4,292 & \\ \text{lower-right:} & 3,267 & \times & 4,292 & \text{new information} \end{array}$$

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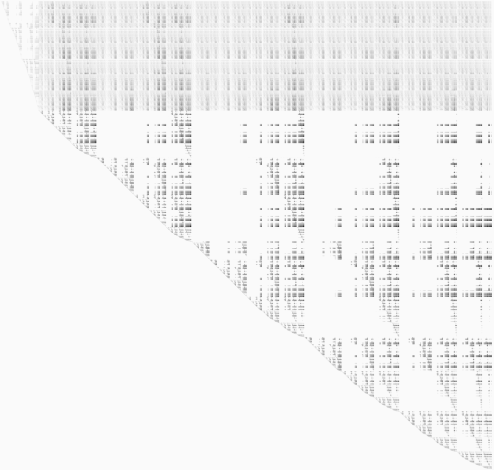

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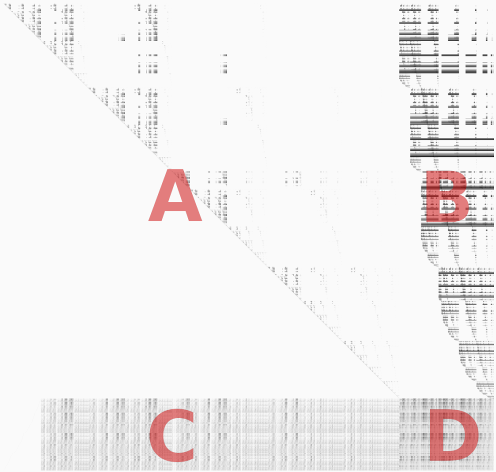
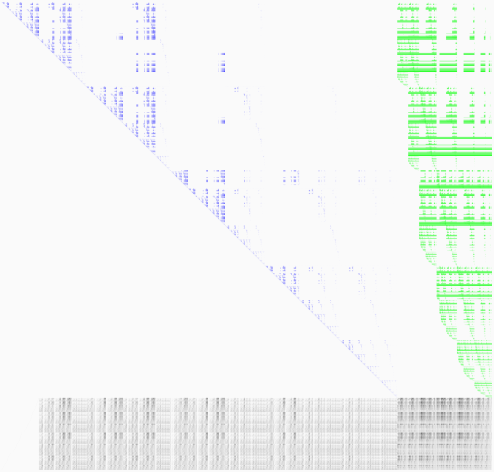


HYBRID MATRIX MULTIPLICATION $A^{-1}B$



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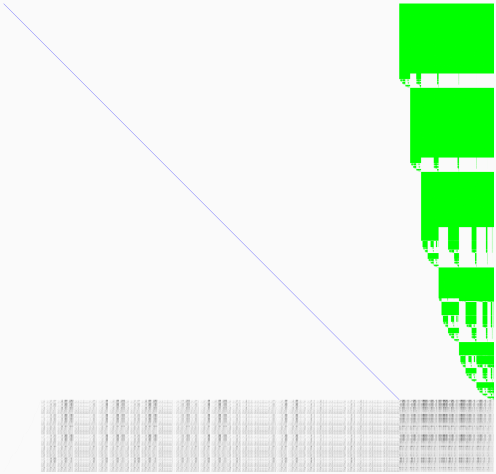
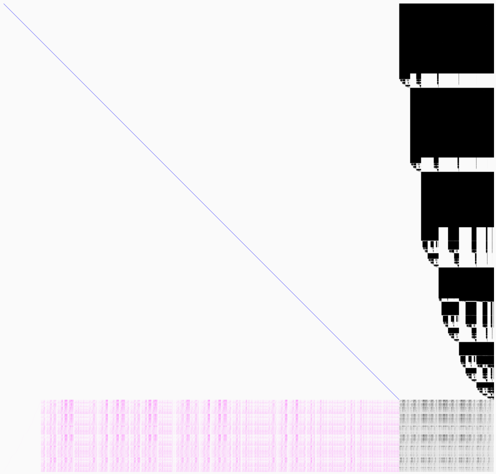
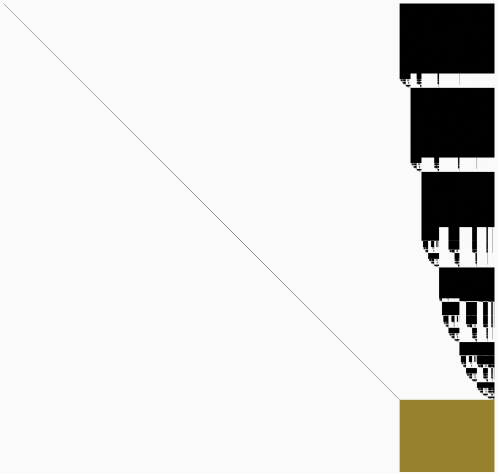
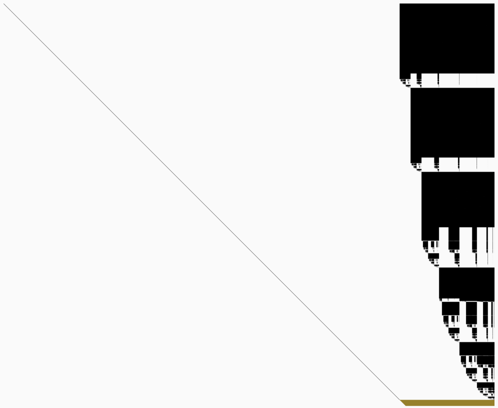




GAUSSIAN ELIMINATION ON D





FEATURES OF GBLA

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`http://hpac.imag.fr/gbla`

Matrices from GB computations have **nonzero entries** often **grouped in blocks**.

Horizontal Pattern If $m_{i,j} \neq 0$ then often $m_{i,j+1} \neq 0$.

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Vertical Pattern If $m_{i,j} \neq 0$ then often $m_{i+1,j} \neq 0$.

EXPLOITING BLOCK STRUCTURES IN MATRICES

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Horizontal Pattern	If $m_{i,j} \neq 0$ then often $m_{i,j+1} \neq 0$.
Vertical Pattern	If $m_{i,j} \neq 0$ then often $m_{i+1,j} \neq 0$.

- Can be used to optimize `AXPY` and `TRSM` operations in FL reduction.

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- Can be used to optimize `AXPY` and `TRSM` operations in FL reduction.
- Horizontal pattern taken care of canonically.
- Need to take care of vertical pattern.

$$
\begin{array}{cc}
A & B \\
\begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & 1 & \cdots & * & * & * & * \\
 & & 1 & \cdots & \boxed{a_{i,j}} & \boxed{a_{i,j+1}} & * \\
 & & & \ddots & & & \vdots \\
 & & & & \boxed{1} & \ddots & * \\
 & & & & & \boxed{1} & * \\
 & & & & & & 1
\end{bmatrix}
&
\begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
* & * & \cdots & * & \cdots & * & * \\
* & * & \cdots & \boxed{b_{i,\ell}} & * & \cdots & * \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
* & * & \cdots & \boxed{b_{k,\ell}} & * & \cdots & * \\
* & * & \cdots & \boxed{b_{k+1,\ell}} & * & \cdots & * \\
* & * & \cdots & * & \cdots & * & *
\end{bmatrix}
\end{array}
$$

Exploiting horizontal and vertical patterns in the TRSM step.

MULTILINE DATA STRUCTURE – AN EXAMPLE

Consider the following two rows:

$$\begin{aligned} \texttt{r1} &= [\ 2\ \ 3\ \ 0\ \ 1\ \ 4\ \ 0\ \ 5\ ], \\ \texttt{r2} &= [\ 1\ \ 7\ \ 0\ \ 0\ \ 3\ \ 1\ \ 2\ ]. \end{aligned}$$

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A sparse vector representation of the two rows would be given by

$$\begin{array}{lcl} \texttt{r1}.\texttt{val} & = & [\ 2\ \ 3\ \ 1\ \ 4\ \ 5\ ], \\ \texttt{r1}.\texttt{pos} & = & [\ 0\ \ 1\ \ 3\ \ 4\ \ 6\ ], \\ & & \\ \texttt{r2}.\texttt{val} & = & [\ 1\ \ 7\ \ 3\ \ 1\ \ 2\ ], \\ \texttt{r2}.\texttt{pos} & = & [\ 0\ \ 1\ \ 4\ \ 5\ \ 6\ ]. \end{array}$$

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A multiline vector representation of **r1** and **r2** is given by

$$\begin{aligned}\texttt{ml}.\texttt{val} &= [\ 2\ \ 1\ \ 3\ \ 7\ \ 1\ \ 0\ \ 4\ \ 3\ \ 0\ \ 1\ \ 5\ \ 2\ ],\\ \texttt{ml}.\texttt{pos} &= [\ 0\ \ 1\ \ 3\ \ 4\ \ 5\ \ 6\ ].\end{aligned}$$

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- Number of initially known pivots (i.e. # rows of A and B) is large compared to # rows of C and D.

NEW ORDER OF OPERATIONS

- Number of initially known pivots (i.e. # rows of A and B) is large compared to # rows of C and D.
- Most time of FL reduction is spent in `TRSM` step $A^{-1}B$.

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- Only interested in D resp. rank of M?

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Table 1: Old matrix format (legacy version)

Size	Length	Data	Description
`uint32_t`	1	`b`	version number
`uint32_t`	1	`m`	# rows
`uint32_t`	1	`n`	# columns
`uint32_t`	1	`p`	prime / field characteristic
`uint64_t`	1	`nnz`	# nonzero entries
`uint16_t`	`nnz`	`data`	entry in matrix
`uint32_t`	`nnz`	`cols`	column index of entry
`uint32_t`	`m`	`rows`	length of rows

Table 2: New matrix format (compressing **data** and **cols**)

Size	Length	Data	Description
uint32_t	1	b	version number + information for data type of **pdata**
uint32_t	1	m	# rows
uint32_t	1	n	# columns
uint32_t	1	p	prime / field characteristic
uint64_t	1	nnz	# nonzero entries
uint16_t	nnz	data	several rows are of type $x_i f_j$
uint32_t	nnz	cols	can be compressed for consecutive elements
uint32_t	m	rows	length of rows
uint32_t	m	pmap	maps rows to **pdata**
uint64_t	1	k	size of compressed **colid**
uint64_t	k	colid	compression of columns: Single column entry masked via $(1 << 31)$; s consecutive entries starting at column c are stored as “c s”
uint32_t	1	pnb	# polynomials
uint64_t	1	pnnz	# nonzero coefficients in polynomials
uint32_t	pnb	prow	length of polynomial / row representation
xinty_t	pnnz	pdata	coefficients of polynomials

Table 3: Storage and time efficiency of the new format

Matrix	Size old	Size new	`gzip`ped old	`gzip`ped new	Time old	Time new
`F4-kat14-mat9`	2.3GB	0.74GB	1.2GB	0.29GB	230s	66s
`F5-kat17-mat10`	43GB	12GB	24GB	5.3GB	4419s	883s

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New format vs. **Old format**

- 1/3rd of memory usage.
- 1/4th of memory usage when compressed with `gzip`.
- Compression 4 – 5 times faster.

SOME BENCHMARKS

All timings in seconds.

Implementation	FL Implementation			GBLA v0.1			GBLA v0.2		
Matrix/Threads:	1	16	32	1	16	32	1	16	32
F5-kat13-mat5	16.7	2.7	2.3	14.5	2.02	1.87	14.5	1.73	1.61
F5-kat13-mat6	27.3	4.15	4.0	23.9	3.08	2.65	25.9	3.03	2.28
F5-kat14-mat7	139	17.4	16.6	142	13.4	10.6	122	11.2	8.64
F5-kat14-mat8	181	24.95	23.1	177	16.9	12.7	158	14.7	10.5
F5-kat15-mat7	629	61.8	55.6	633	55.1	38.2	553	46.3	30.7
F5-kat16-mat6	1,203	110	83.3	1,147	98.7	69.9	988	73.9	49.0
F5-mr-9-10-7-mat3	591	70.8	71.3	733	57.3	37.9	747	52.8	33.2
F5-cyclic-10-mat20				2,589	274	209	2,074	171	152
F5-cyclic-10-sym-mat17				2,463	465	405	2,391	275	245

GBLA VS. MAGMA V2.20-10

All timings in seconds.

Implementation	Magma	GBLA v0.1			GBLA v0.2		
Matrix/Threads:	1	1	16	32	1	16	32
F4-kat12-mat9	11.2	11.4	1.46	1.60	11.3	1.40	1.40
F4-kat13-mat2	0.94	1.18	0.38	0.61	1.11	0.26	0.33
F4-kat13-mat3	9.33	11.0	1.70	3.10	8.51	1.07	1.13
F4-kat13-mat9	168	165	16.0	11.8	114	9.74	6.83
F4-kat14-mat8	2,747	2,545	207	165	1,338	104	65.8
F4-kat15-mat7	10,345	9,514	742	537	4,198	298	195
F4-kat15-mat8	13,936	12,547	961	604	6,508	470	283
F4-kat15-mat9	24,393	22,247	1,709	1,256	10,923	779	450
F4-rand16-d2-2-mat6		4,902	375	219	3,054	224	133
F4-rand16-d2-3-mat8		48,430	3,473	2,119	26,533	1,782	1,027
F4-rand16-d2-3-mat9			6,956	4,470		3,214	1,776
F4-rand16-d2-3-mat10[1]			9,691	6,223		3,820	1,972

Note that Magma generates slightly bigger matrices for the given examples.

[1]Reconstruction fails due to memory consumption

OUTLOOK

DIFFERENT APPROACHES

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- Creation of a new open source plain C library **GBTOOLS**.
- Deeper investigation on **parallelization on networks**.
- First steps exploiting **heterogeneous CPU/GPU platforms** for GBLA.

REFERENCES

Buchberger, B.
Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal, 1965.
PhD thesis, Universtiy of Innsbruck, Austria

Buchberger, B.
A criterion for detecting unnecessary reductions in the construction of Gröbner bases, 1979.
EUROSAM '79, An International Symposium on Symbolic and Algebraic Manipulation

Buchberger, B.
Gröbner Bases: An Algorithmic Method in Polynomial Ideal Theory, 1985.
Multidimensional Systems Theory, D. Reidel Publication Company

Eder, C. and Faugère, J.-C.
A survey on signature-based Groebner basis algorithms, 2014.
http://arxiv.org/abs/1404.1774

Faugère, J.-C.
A new efficient algorithm for computing Gröbner bases (F4), 1999.
Journal of Pure and Applied Algebra

Faugère, J.-C.
A new efficient algorithm for computing Gröbner bases without reduction to zero (F5), 2002.
Proceedings of the 2002 international symposium on Symbolic and algebraic computation

Faugère, J.-C. and Lachartre, S.
Parallel Gaussian Elimination for Gröbner bases computations in finite fields, 2010.
Proceedings of the 4th International Workshop on Parallel and Symbolic Computation

Gebauer, R. and Möller, H. M.
On an installation of Buchberger's algorithm, 1988.
Journal of Symbolic Computation

THANK YOU!

COMMENTS? QUESTIONS?