Modular Techniques in Computational Algebraic Geometry

Janko Boehm
joint with W. Decker, C. Fieker, S. Laplagne, G. Pfister

Technische Universität Kaiserslautern

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- Goal:
  General reconstruction scheme for algorithms in commutative algebra, algebraic geometry, number theory.

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- Normalization
- Local-to-global algorithm for adjoint ideals
- Modular version and verification

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How to obtain a rational number from $\overline{22684}$?

Rational reconstruction

Theorem (Kornerup, Gregory, 1983)

The **Farey map**

$$\left\{ \tfrac{a}{b} \in \mathbb{Q} \;\middle|\; \begin{matrix} \gcd(a,b) = 1 \\ \gcd(b,N) = 1 \end{matrix} \quad |a|\,,|b| \leq \sqrt{(N-1)/2} \right\} \longrightarrow \mathbb{Z}/N$$

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Definition

A prime p is called **bad** if the result over $\mathbb{Q}$ does not reduce modulo p to the result over $\mathbb{Z}/p$.

Bad primes in Gröbner basis computations

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$$\begin{aligned} p \text{ does not divide any lead coefficient in } G_{\mathbb{Z}} &\iff \mathrm{LM}\,G = \mathrm{LM}\,G(p) \\ &\iff G_p = G(p) \end{aligned}$$

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that is, p is not bad.

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w.r.t *lp* is

$$264627y^{39} + \ldots,$$

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and LM G = LM $G(p)$ for all primes p except

$$p = 3, 5, 11, 809, 65179, 531264751, 431051934846786628615463393.$$

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- Type 5: otherwise.

Example of type 5 bad prime

For ideal $I \subset \mathbb{Q}[X]$ and prime p define $I_p = (I \cap \mathbb{Z}[X])_p$.

Example

Consider the algorithm $I \mapsto \sqrt{I + \mathrm{Jac}(I)}$ for

$$I = \langle x^6 + y^6 + 7x^5z + x^3y^2z - 31x^4z^2 - 224x^3z^3 + 244x^2z^4 + 1632xz^5 + 576z^6 \rangle$$

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$$U(0) = \sqrt{I + \mathrm{Jac}(I)} = \langle y, x - 4z \rangle \cap \langle y, x + 6z \rangle$$

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Hence

$$U(0)_5 \neq U(5)$$

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Proof.

Let $\lambda = (x, y)$, $\mu = (c, d) \in \Lambda$ with $x^2 + y^2, c^2 + d^2 < N$. Then $y\mu - d\lambda = (yc - xd, 0) \in \Lambda$, so $N | (yc - xd)$. By Cauchy–Schwarz $|yc - xd| < N$, hence $yc = xd$. □

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All $(x, y) \in \Lambda$ with $x^2 + y^2 < N$ are collinear.

Proof.

Let $\lambda = (x, y)$, $\mu = (c, d) \in \Lambda$ with $x^2 + y^2, c^2 + d^2 < N$. Then $y\mu - d\lambda = (yc - xd, 0) \in \Lambda$, so $N | (yc - xd)$. By Cauchy–Schwarz $|yc - xd| < N$, hence $yc = xd$. □

Now suppose

$$N = N' \cdot M$$

with $\gcd(N', M) = 1$.

Error tolerant reconstruction

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then $(aM, bM) \in \Lambda$. *So if*

$$(a^2 + b^2)M < N',$$

then (by the lemma)

$$\frac{x}{y} = \frac{a}{b} \quad \textit{for all } (x, y) \in \Lambda \textit{ with } (x^2 + y^2) < N$$

and such vectors exist.

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and such vectors exist. Moreover, if $\gcd(a, b) = 1$ *and* (x, y) *is a shortest vector* $\neq 0$ *in* Λ, *we also have* $\gcd(x, y) | M$.

Error tolerant reconstruction via Gauss-Lagrange

Hence, if $N' \gg M$, the Gauss-Lagrange-Algorithm for finding a shortest vector $(x, y) \in \Lambda$ gives $\frac{a}{b}$ independently of t, provided $x^2 + y^2 < N$.

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Algorithm (Error tolerant reconstruction)

Input: *N and r.*

Output: *$\frac{a}{b}$ or false.*

1: $(a_0, b_0) := (N, 0)$, $(a_1, b_1) := (r, 1)$, $i := -1$

2: **repeat**

3: $\quad i = i + 1$

4: $\quad (a_{i+2}, b_{i+2}) = (a_i, b_i) - \left\lfloor \frac{\langle (a_i, b_i), (a_{i+1}, b_{i+1}) \rangle}{\| (a_{i+1}, b_{i+1}) \|^2} \right\rceil (a_{i+1}, b_{i+1})$

5: **until** $a_{i+2}^2 + b_{i+2}^2 \geq a_{i+1}^2 + b_{i+1}^2$

6: **if** $a_{i+1}^2 + b_{i+1}^2 < N$ **then**

7: $\quad$ **return** $\frac{a_{i+1}}{b_{i+1}}$

8: **else**

9: $\quad$ **return** `false`

Reconstruction via Gauss-Lagrange

Example

We reconstruct $\frac{13}{12}$ from

$$\overline{22684} \in \mathbb{Z}/38885$$

by determining a shortest vector in the lattice

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$$\begin{aligned}
(38885, 0) &= 2 \cdot (22684, 1) + (-6483, -2), \\
(22684, 1) &= -3 \cdot (-6483, -2) + (3235, -5), \\
(-6483, -2) &= 2 \cdot (3235, -5) + (-13, -12), \\
(3235, -5) &= -134 \cdot (-13, -12) + (1493, -1613).
\end{aligned}$$

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Now introduce an error in the modular results:

$$\begin{array}{ccccccccc} \mathbb{Z}/5 & \times & \mathbb{Z}/7 & \times & \mathbb{Z}/11 & \times & \mathbb{Z}/101 & \cong & \mathbb{Z}/38885 \\ (\,\overline{4} & , & \overline{4} & , & \overline{2} & , & \overline{60}\,) & \mapsto & \overline{22684} \end{array}$$

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Error tolerant reconstruction computes

$$\begin{aligned}
(38885, 0) &= 84 \cdot (464, 1) + (-91, -84), \\
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Note that

$$(13^2 + 12^2) \cdot 7 = 2191 < 5555 = 5 \cdot 11 \cdot 101.$$

General reconstruction scheme

Setup: For ideal $I \subset \mathbb{Q}[X]$ compute ideal (or module) $U(0)$ associated to I by deterministic algorithm.

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Theorem (BDFP, 2015)

If the bad primes form a Zariski closed true subset of $\operatorname{Spec} \mathbb{Z}$, then this algorithm terminates with the correct result.

Normalization

Setup: $A = K[X]/I$ domain.

Definition

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Example

Curve $I = \langle x^3 + x^2 - y^2 \rangle \subset K[x, y]$

$$\begin{array}{ccccc} A = K[x,y]/I & \cong & K[t^2-1, t^3-t] & \subset & K[t] \cong \overline{A} \\ \overline{x} & \mapsto & t^2 - 1 & & \\ \overline{y} & \mapsto & t^3 - t & & \end{array}$$

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As an A-module $\overline{A} = \left\langle 1, \frac{\overline{y}}{\overline{x}} \right\rangle$.

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we get a chain of extensions of reduced Noetherian rings

$$A = A_0 \subset \cdots \subset A_i \subset \cdots \subset A_m = A_{m+1}.$$

Terminates since A is Noetherian.

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Lemma

$N(A_i) \subset V(\sqrt{JA_i})$

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and

$$\overline{A} = B_1 + \ldots + B_r.$$

We call B_i *the* **minimal local contribution** *to* $\overline{A}$ *at* P_i.

Adjoint ideals

Setup: $\Gamma \subset \mathbb{P}^r$ integral, non-degenerate projective curve, $\pi : \overline{\Gamma} \to \Gamma$ normalization map, $I(\Gamma) \subsetneq I \subset k[x_0, ..., x_r]$ saturated homogeneous ideal.

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Definition

I is an **adjoint ideal** of Γ if $\overline{\varrho_m}$ surjective for $m \gg 0$.

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$$h^0\left(\Gamma, \mathcal{F}\right) = \sum_{P \in \mathrm{Sing}(\Gamma)} \ell(I_P \overline{\mathcal{O}_{\Gamma,P}}/I_P) \qquad \Longrightarrow$$

Theorem

$$I \text{ adjoint} \iff I_P\overline{\mathcal{O}_{\Gamma,P}} = I_P \text{ for all } P \in \mathrm{Sing}(\Gamma).$$

Conductor is largest ideal with this property.

Adjoint ideals

Definition

Gorenstein adjoint ideal is the unique largest homogeneous ideal $\mathfrak{G} \subset K[x_0, \ldots, x_r]$ with

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If Γ is plane rational of degree n then $\mathfrak{G}_{n-2}$ maps Γ to rational normal curve of degree $n-2$ in $\mathbb{P}^{n-2}$.

Adjoint ideals

Definition

Gorenstein adjoint ideal is the unique largest homogeneous ideal $\mathfrak{G} \subset K[x_0, \ldots, x_r]$ with

$$\mathfrak{G}_P = \mathcal{C}_{\mathcal{O}_{\Gamma,P}} \text{ for all } P \in \operatorname{Sing}(\Gamma).$$

Applications:

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Brill-Noether-Algorithm for computing Riemann-Roch spaces.

Example

Minimal generators of $\mathfrak{G}$ for rational curve of degree 5:

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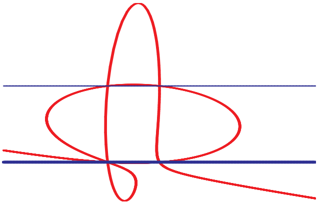

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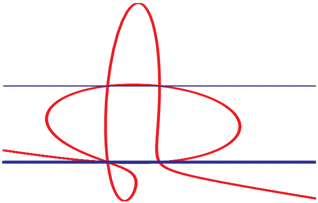

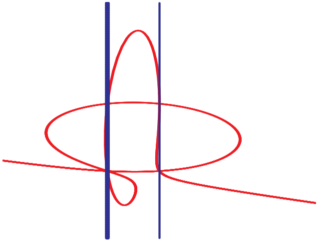

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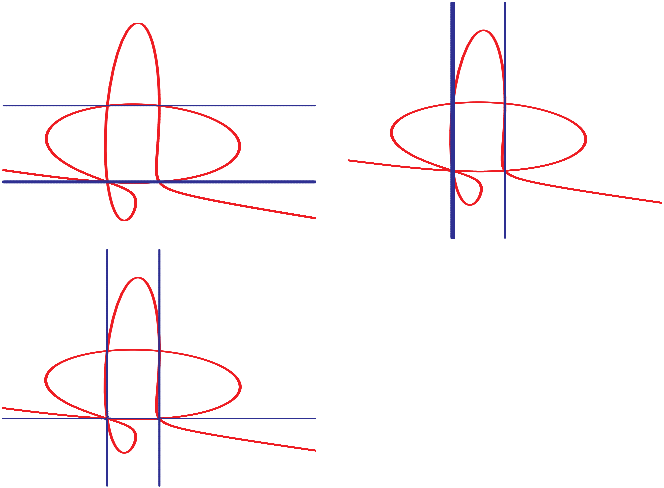

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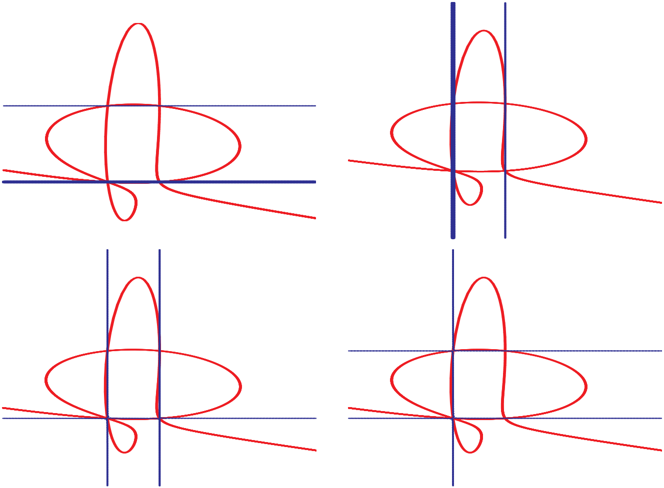

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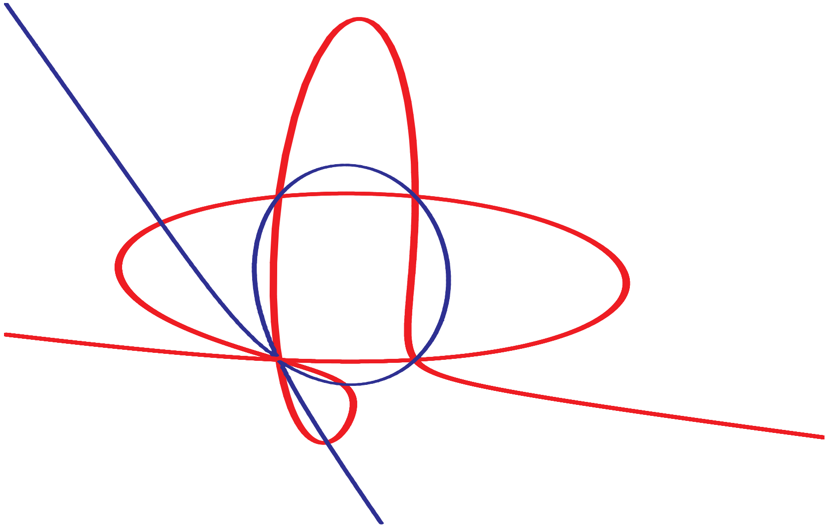

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Definition

The **local adjoint ideal** of Γ at $P \in \operatorname{Sing}\Gamma$ is the largest homogeneous ideal $\mathfrak{G}(P) \subset k[x_0, \ldots, x_r]$ with

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Algorithm (BDLP, 2015)

If $\frac{1}{d}U$ is the minimal local contribution at P then

$$\mathfrak{G}(P) = (d : U)^h$$

Special types of singularities

If $\Gamma \subset \mathbb{P}^2$ has a singularity of type A_n at $P = (0:0:1)$, then given by

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Similar results for D_n, E_n and other singularities in Arnold's list.

Example

$f = x^4 - y^2 + x^5$ with A_3 singularity. Then $\mathfrak{G}(P) = \langle x^2, y \rangle$.

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Theorem (BDLP, 2015, corollary to Lipman, 2006)

$$\delta(\Gamma) \leq \delta(\Gamma_p)$$

and δ*-constant flat family admits a simultaneous normalization.*

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then

$$\deg \Delta(I) = \deg \Delta(I_p) = (\deg \Gamma) \cdot m - \widetilde{d}(g_p)$$
$$\delta(\Gamma) = \delta(\Gamma_p)$$

and I is an adjoint ideal.

Timings in SINGULAR

Plane curve f_n of degree n with $\binom{n-1}{2}$ singularities of type A_1.

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	parallel	probablisitic	f_5		f_6		f_7	
locNormal			2.1		56		-	
Maple-IB			5.1		47		318	
LA			98		4400		-	
IQ			1.3		54		3800	
locIQ	■		1.3	(1)	54	(1)	3800	(1)
ADE	■		.18	(1)	1.2	(1)	49	(1)
modLocIQ			6.4	[33]	19	[53]	150	[75]
		■	6.2	[33]	18	[53]	104	[75]
	■		.36	(74)	1.6	(153)	51	(230)
	■	■	.21	(74)	0.48	(153)	5.2	(230)

[primes] (cores)

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Plane curve $f_{n,d}$ of degree d with one singularity of type D_n.
Curves h_1, h_2 of degree 20 and 28 in $\mathbb{P}^5$.

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Curves h_1, h_2 of degree 20 and 28 in $\mathbb{P}^5$.

	parallel	probabilistic	$f_{50,500}$		$f_{400,500}$		h_1		h_2	
locNormal			.67		4.9		21		-	
Maple-IB			1830		-		N/A		N/A	
LA			-		-		N/A		N/A	
IQ			.67		5.0		30		-	
locIQ	■		.67	(1)	5.0	(1)	7.5	(6)	-	
ADE	■		.58	(1)	5.0	(1)	N/A		N/A	
modLocIQ		■	1.5	[2]	24	[2]	27	[3]	2600	[5]
	■	■	.77	(2)	17	(2)	4.0	[27]	59	(69)

[primes] (cores)

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