The functional equation for L-functions of hyperelliptic curves

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Setup

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g	genus $g = g(Y)$
N	conductor, $N = \prod_p p^{f_p}$

Functional Equation

Conjectured functional equation

$$\Lambda(Y,s) = \pm\Lambda(Y,2-s) \qquad \textbf{(FEq)}$$

where $\Lambda(Y,s) = \sqrt{N}^s \cdot (2\pi)^{-gs} \cdot \Gamma(s)^g \cdot L(Y,s)$

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- $\sum_n \frac{a_n}{n^s} = \prod_p L_p(Y,s) \rightsquigarrow$ need sufficiently many local factors L_p

Good and bad primes

Good, bad and semistable reduction at a prime

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- $f_p = 0$ $\rightsquigarrow$ no contribution to conductor N

Good and bad primes ctd.

Bad p:

- L_p, f_p depend on Galois representation on $H^1_{\text{et}}(Y_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$

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Next: Class of hyperelliptic curves over $\mathbb{Q}$
with semistable reduction everywhere

Hyperelliptic curves with semistable red. at all p

Hyperelliptic setup ($p \neq 2$ case)

$g, h \in \mathbb{Z}_p[x]$, g monic, $\deg g = 2g(Y) + 1$, $\deg h \leq g(Y)$, $\gcd(f, f') = 1$

$$Y: \quad y^2 + h(x)y = g(x)$$

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$\boldsymbol{p = 2}$ **:** A little more work

Computations at bad primes $p \neq 2$

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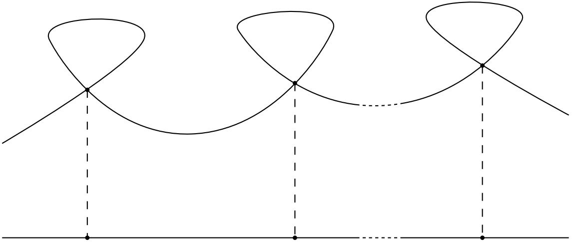

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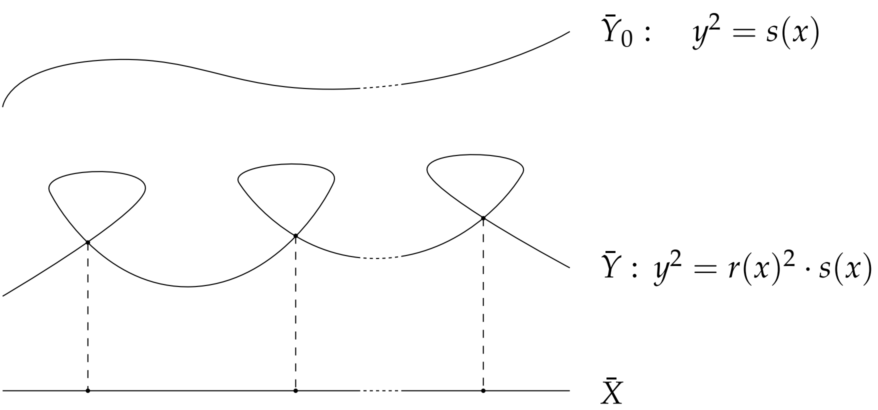

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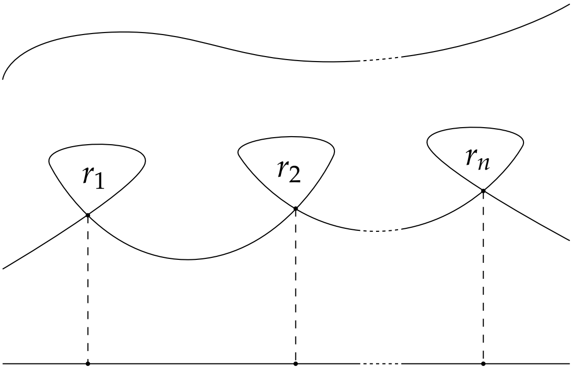

$\bar{Y}_0 : \quad y^2 = s(x)$

$\bar{Y} : y^2 = r(x)^2 \cdot s(x)$

$r = \prod_i r_i \in \mathbb{F}_p[x]$

$\bar{X}$

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Point counting (good/bad p): Just for $p^k \leq M$ where $M \sim \sqrt{N}$

Example

$$Y: y^2 + h(x)y = g(x)$$

$$g = x^7 + x^6 + 2x^5 + 2x^4 + 2x^3 - 1$$

$$h = -x^3 + x^2 + x + 2$$

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Example ctd.

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Verification of (**FEq**) via `Dokchitser` package in *sage*: ✓

Overview and loose ends

Hyperelliptic curves

- All L_p, f_p computable for $g = 2, 3, 4, 5, 6$ with reasonable effort

  Limitation: point count for good $p \sim \sqrt{N} = \prod_{\text{bad } p} p^{f_p/2}$

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- Find algorithms for L_p and f_p for wider range of curves
- Find (all) curves of certain class with small N (in preparation)

Thank you!

[BW15] *Computing L-functions and semistable reduction of superelliptic curves,* Irene Bouw and Stefan Wewers. To appear in Glasgow Math. J.

[BBW15] *The functional equation for L-functions of hyperelliptic curves,* Michel Börner, Irene I. Bouw, Stefan Wewers, arXiv:1504.00508

[B15] *Examples of [BBW15],* M.B., `https://www.uni-ulm.de/index.php?id=64504`

[D04] *Computing special values of motivic L-functions,* Tim Dokchitser, Exper. Math. 13, No. 2 (2004), 137-149