Binomial edge ideals and determinantal facet ideals

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Binomial edge ideals

Let $G$ be a finite simple graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. Associated to $G$ is a binomial ideal

$$J_G = (f_{ij} : i < j, \ \{v_i, v_j\} \in E(G)),$$

in $S = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$, called the binomial edge ideal of $G$, in which $f_{ij} = x_i y_j - x_j y_i$.

It could be seen as the ideal generated by a collection of 2-minors of a $(2 \times n)$-matrix whose entries are all indeterminates.
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It could be seen as the ideal generated by a collection of 2-minors of a $(2 \times n)$-matrix whose entries are all indeterminates.
By $<$, we mean the lexicographic order induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$.

**Herzog - Hibi - Hreinsdóttir - Kahle - Rauh (2010)**

Let $G$ be a graph. Then $\text{in}_{<} J_G$ is a squarefree monomial ideal. In particular, $J_G$ is a radical ideal.
Let $G$ be a graph $[n]$, and let $G_1, \ldots, G_{c(T)}$ be the connected component of $G[n] \setminus T$, the induced subgraph of $G$ on $[n] \setminus T$. For each $G_i$ we denote by $\tilde{G}_i$ the complete graph on the vertex set $V(G_i)$. For each subset $T \subset [n]$ a prime ideal $P_T(G)$ is defined as

$$P_T(G) = \left( \bigcup_{i \in T} \{x_i, y_i\}, J_{\tilde{G}_1}, \ldots, J_{\tilde{G}_{c(T)}} \right).$$
Let $G$ be a graph $[n]$. Then $J_G = \bigcap_{T \subseteq [n]} P_T(G)$.

Let $G$ be a graph $[n]$. Then $P_T(G)$ is a minimal prime ideal of $J_G$ if and only if $T = \emptyset$, or each $i \in T$ is a cut point of the graph $G([n] \setminus T) \cup \{i\}$. 
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**Corollary**

$J_G$ is a prime ideal if and only if all connected components of $G$ are complete graphs.

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Corollary

Let $G$ be a graph $[n]$. Then $\text{height}_{P_T}(G) = |T| + (n - c(T))$ and

$$\dim S/J_G = \max\{(n - |T|) + c(T) : T \subset [n]\}.$$ 

In particular, $\dim S/J_G \geq n + c$, where $c$ is the number of connected components of $G$. 

The following conditions are equivalent:

(1) The generators $f_{ij}$ of $J_G$ form a quadratic Gröbner basis.

(2) For all edges $\{i, j\}$ and $\{k, l\}$ with $i < j$ and $k < l$ one has $\{j, l\} \in E(G)$ if $i = k$, and $\{i, k\} \in E(G)$ if $j = l$. 
A graph $G$ is said to be closed with respect to the given labeling of the vertices, if $G$ satisfies conditions of previous theorem, and a graph $G$ with vertex set $V(G) = \{v_1, \ldots, v_n\}$ is said to be closed, if its vertices can be labeled by the integer $1, 2, \ldots, n$ such that for this labeling $G$ is closed.
Closed graphs

$C_5$ is not a closed graph.
Closed graphs

$P_n$ is a closed graph.
Ene - Herzog - Hibi (2010)

The following conditions are equivalent:

(1) $G$ is closed.

(2) There exists a labeling of $G$ such that all facets of the clique complex of $G$ are intervals.
Ene - Herzog - Hibi (2010)

Let $G$ be a closed graph with Cohen-Macaulay binomial edge ideal. Then $\beta_{ij}(J_G) = \beta_{ij}(\text{in}_{<}(J_G))$ for all $i, j$.

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Suppose $I$ is a homogeneous ideal of $R$ whose generators all have degree $d$. Then $I$ has a linear resolution if for all $i \geq 0$, $\beta_{i,j}(I) = 0$ for all $j \neq i + d$. 
Let $G$ be a graph with no isolated vertices. Then the following conditions are equivalent:

1. $J_G$ has a linear resolution.
2. $J_G$ is linearly presented.
3. $\text{in}_<(J_G)$ has a linear resolution.
4. $G$ is a complete graph.
Let $I$ be a homogeneous ideal of $S$ whose generators all have degree $d$. Then $I$ has a $d$-pure resolution (or pure resolution) if its minimal graded free resolution is of the form

$$0 \to S(-d_p)^{\beta_p(I)} \to \cdots \to S(-d_1)^{\beta_1(I)} \to I \to 0,$$

where $d = d_1$. 
If $G$ is a complete bipartite graph, then $J_G$ has a pure resolution.
Let $G$ be a graph with no isolated vertices. Then $J_G$ has a pure resolution if and only if $G$ is a:

(1) complete graph, or
(2) complete bipartite graph, or
(3) disjoint union of some paths.
Let $G$ be a graph on $[n]$, and let $\ell$ be the length of the longest induced path in $G$. Then

$$\text{reg}(J_G) \geq \ell + 1.$$
Regularity

Denoted $c(G)$ we mean the number of maximal cliques of $G$.

Kiani - SM (2012)
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Let $G$ be a graph. Then $\text{reg}(J_G) \leq c(G) + 1$. 
Ene - Zarojanu (2014)

Let $G$ be a block graph. Then $\text{reg}(J_G) \leq c(G) + 1$. 
Let $G$ be a closed graph with connected components $G_1, \ldots, G_r$. Then
\[
\text{reg}(J_G) = \text{reg}(\text{in}_{<}(J_G)) = \ell_1 + \cdots + \ell_r + 1,
\]
where $\ell_i$ is the length of the longest induced path of $G_i$. 
Kiani - SM (2015)

Let $G_1$ and $G_2$ be graphs on $[n_1]$ and $[n_2]$, respectively, not both complete. Then

$$\text{reg}(J_{G_1 \ast G_2}) = \max\{\text{reg}(J_{G_1}), \text{reg}(J_{G_2}), 3\}.$$ 

Corollary

Let $G$ be a complete $t$-partite graph which is not complete. Then

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It was proved before for bipartite graphs by Schenzel and Zafar.
Let $G_1$ and $G_2$ be graphs on $[n_1]$ and $[n_2]$, respectively, not both complete. Then

$$\text{reg}(J_{G_1 * G_2}) = \max\{\text{reg}(J_{G_1}), \text{reg}(J_{G_2}), 3\}.$$  

**Corollary**

Let $G$ be a complete $t$-partite graph which is not complete. Then $\text{reg}(J_G) = 3$.

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Matsuda - Murai (2013)

Let $G$ be a graph on $n$ vertices. Then $\text{reg}(J_G) \leq n$.

Conjecture (Matsuda - Murai (2013))

Let $G \neq P_n$ be a graph on $n$ vertices. Then $\text{reg}(J_G) \leq n - 1$. 
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Matsuda and Murai’s Conjecture

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Let $C_n$ be an $n$-cycle. Then $\text{reg}(J_{C_n}) = n - 1$.

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Mohammadi - Sharifan (2014)

Let $G$ be a graph and $e = \{i, j\}$ be an edge of $G$. Then

$$J_{G \setminus e} : f_e = J_{(G \setminus e)_e} + I_G,$$

where

$$I_G = (g_{P,t} : P : i, i_1, \ldots, i_s, j \text{ and } 0 \leq t \leq s),$$

$g_{P,0} = x_{i_1} \cdots x_{i_s}$ and $g_{P,t} = y_{i_1} \cdots y_{i_t} x_{i_{t+1}} \cdots x_{i_s}$ for every $1 \leq t \leq s$. 
Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring. We view $S$ as a standard graded $K$-algebra by assigning to each $x_i$ the degree 1. A graded complex

$$
\mathcal{G} : \cdots \to G_2 \to G_1 \to G_0 \to 0
$$

of finitely generated graded free $S$-modules is called a linear complex (with initial degree $d$) if for all $i$, $G_i = S(-i - d)^{b_i}$ for suitable integers $b_i$. 
Let $M$ be a finitely generated graded $S$-module, and let $d$ be the initial degree of $M$, and let $(\mathcal{F}, \partial)$ be the minimal graded free resolution of $M$ with $F_i = \bigoplus_j S(-j)^{\beta_{i,j}}$. Note that $\beta_{ij} = 0$ for all pairs $(i,j)$ with $j < i + d$.

Let $F_{i_{\text{lin}}}^i$ be the direct summand $S(-i - d)^{\beta_{i,i+d}}$ of $F_i$. It is obvious that $\partial(F_{i_{\text{lin}}}^i) \subset F_{i-1_{\text{lin}}}^i$ for all $i > 0$. 
Let $M$ be a finitely generated graded $S$-module, and let $d$ be the initial degree of $M$, and let $(F, \partial)$ be the minimal graded free resolution of $M$ with $F_i = \bigoplus_j S(-j)^{\beta_{i,j}}$. Note that $\beta_{ij} = 0$ for all pairs $(i, j)$ with $j < i + d$.

Let $F_{i}^{\text{lin}}$ be the direct summand $S(-i - d)^{\beta_{i,i+d}}$ of $F_i$. It is obvious that $\partial(F_{i}^{\text{lin}}) \subset F_{i-1}^{\text{lin}}$ for all $i > 0$. 
Thus
\[ F^{\text{lin}} : \cdots \rightarrow F_2^{\text{lin}} \rightarrow F_1^{\text{lin}} \rightarrow F_0^{\text{lin}} \rightarrow 0 \]
is a subcomplex of $F$, called the linear strand of the resolution of $M$.

Obviously, $F^{\text{lin}}$ is a linear complex.
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Obviously, \( F^\text{lin} \) is a linear complex.
The linear strand

Denoted by \((f_0(\Delta), f_1(\Delta), \ldots, f_d(\Delta))\) is the \(f\)-vector of a \(d\)-dimensional simplicial complex \(\Delta\).

**Conjecture (Kiani - SM (2014))**

Let \(G\) be a graph. Then \(\beta_{i,i+2}(J_G) = (i + 1)f_{i+1}(\Delta(G))\), where \(\Delta(G)\) is the clique complex of \(G\).
A clutter $C$ on the vertex set $[n]$ is a collection of subsets of $[n]$ with no containment between its elements. An element of $C$ is called a circuit. If all circuits of $C$ have the same cardinality $m$, then $C$ is called an $m$-uniform clutter.

A clique of an $m$-uniform clutter $C$ is a subset $\sigma$ of $[n]$ such that each $m$-subset of $\sigma$ is a circuit of $C$. We denote by $\Delta(C)$ the simplicial complex whose faces are the cliques of $C$ which is called the clique complex of $C$. 
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An $m$-uniform clutter is called complete if its clique complex is a simplex.
Let $C$ be an $m$-uniform clutter on $[n]$. To each circuit $\tau \in C$ with $\tau = \{j_1, \ldots, j_m\}$ and $1 \leq j_1 < j_2 < \cdots < j_m \leq n$ we assign the $m$-minor $m_\tau$ of $X = (x_{ij})$ which is determined by the columns $1 \leq j_1 < j_2 < \cdots < j_m \leq n$.

Denoted by $J_C$ is the ideal in $S = K[x_{ij} : 1 \leq i \leq m, 1 \leq j \leq n]$ which is generated by the minors $m_\tau$ with $\tau \in C$. This ideal is called the determinantal facet ideal of $C$. 
Let $C$ be an $m$-uniform clutter on $[n]$. To each circuit $\tau \in C$ with $\tau = \{j_1, \ldots, j_m\}$ and $1 \leq j_1 < j_2 < \cdots < j_m \leq n$ we assign the $m$-minor $m_\tau$ of $X = (x_{ij})$ which is determined by the columns $1 \leq j_1 < j_2 < \cdots < j_m \leq n$.

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In the case that $C$ is a 2-uniform clutter, $C$ may be viewed as a graph $G$, and hence $J_C = J_G$. 
Let $G$ be a finite linear complex with initial degree $d$. Then the following conditions are equivalent:

(1) $G$ is the linear strand of a finitely generated graded $S$-module with initial degree $d$.

(2) $H_i(G)_{i+d+j} = 0$ for all $i > 0$ and for $j = 0, 1$. 
Let $F$ and $G$ be free $S$-modules of rank $m$ and $n$, respectively, with $m \leq n$, and let $\varphi : G \to F$ be an $S$-module homomorphism.

We choose a basis $f_1, \ldots, f_m$ of $F$ and a basis $g_1, \ldots, g_n$ of $G$. Let $\varphi(g_j) = \sum_{i=1}^{m} \alpha_{ij} f_i$ for $j = 1, \ldots, n$. The matrix $\alpha = (\alpha_{ij})$ describing $\varphi$ with respect to these bases is an $(m \times n)$-matrix with entries in $S$. 
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The ideal of $m$-minors of this matrix is denoted $I_m(\varphi)$. It is known that if $\text{grade } I_m(\varphi) = n - m + 1$, then the so-called Eagon-Northcott complex provides a free resolution of $I_m(\varphi)$.
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Denote by $S(F)$ is the symmetric algebra of $F$. The complex

\[ C(\varphi) : 0 \to \bigwedge^n G \otimes S_{n-m}(F)^* \to \cdots \to \bigwedge^m G \otimes S_0(F)^* \to 0, \]

is called the Eagon-Northcott complex.
We set $C_i(\varphi) = \bigwedge^{m+i} G \otimes S_i(F)^*$ and $b(\sigma; a) = g_\sigma \otimes f(a)$, where $g_\sigma = g_{j_1} \wedge \cdots \wedge g_{j_{m+i}}$ for $\sigma = \{j_1 < j_2 < \cdots < j_{m+i}\}$, and $f(a)$ is the dual of $f^a = f_1^{a_1} f_2^{a_2} \cdots f_m^{a_m}$ with $a \in \mathbb{Z}_{\geq 0}^m$ and $|a| = a_1 + \cdots + a_m = i$. Moreover, we set $f(a) = 0$ if $a_i < 0$ for some $i$.

Then the elements $b(\sigma; a)$ form a basis of $C_i(\varphi)$, and

$$
\partial(b(\sigma; a)) = \sum_{k=1}^{m} \sum_{\ell=1}^{m} (-1)^{k+1} \alpha_{\ell j_k} b(\sigma \setminus \{j_k\}; a - e_\ell).
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Here $e_1, \ldots, e_m$ is the canonical basis of $\mathbb{Z}^m$. 

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$$\partial(b(\sigma; a)) = \sum_{k=1}^{m+i} \sum_{\ell=1}^m (-1)^{k+1} \alpha_{\ell j_k} b(\sigma \setminus \{j_k\}; a - e_{\ell}).$$

Here $e_1, \ldots, e_m$ is the canonical basis of $\mathbb{Z}^m$. 
Let $\Delta$ be a simplicial complex on $[n]$. We denote $C_i(\Delta; \varphi)$ the free submodule of $C_i(\varphi)$ generated by all $b(\sigma; a)$ such that $\sigma \in \Delta$ with $|\sigma| = m + i$, and $a \in \mathbb{Z}_{\geq 0}^m$ with $|a| = i$.

Since $\partial(b(\sigma; a)) \in C_{i-1}(\Delta; \varphi)$ for all $b(\sigma; a) \in C_i(\Delta; \varphi)$, we obtain the subcomplex

$$C(\Delta; \varphi) : 0 \rightarrow C_{n-m}(\Delta; \varphi) \rightarrow \cdots \rightarrow C_1(\Delta; \varphi) \rightarrow C_0(\Delta; \varphi) \rightarrow 0$$

of $C(\varphi)$ which we call the generalized Eagon-Northcott complex attached to the simplicial complex $\Delta$ and the module homomorphism $\varphi : G \rightarrow F$. 
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Let $X$ be an $(m \times n)$-matrix of indeterminates $x_{ij}$, and let $S$ be the polynomial ring over a field $K$ in the variables $x_{ij}$. Moreover, let $\varphi : G \rightarrow F$ be the $S$-module homomorphism of free $S$-modules given by the matrix $X$.

Now we give a $(\mathbb{Z}^m \times \mathbb{Z}^n)$-grading to the polynomial ring $S$, by setting $m\text{deg}(x_{ij}) = (e_i, \varepsilon_j)$ where $e_i$ is the $i$-th canonical basis vector of $\mathbb{Z}^m$ and $\varepsilon_j$ is the $j$-th canonical basis vector of $\mathbb{Z}^n$. 
Let $X$ be an $(m \times n)$-matrix of indeterminates $x_{ij}$, and let $S$ be the polynomial ring over a field $K$ in the variables $x_{ij}$. Moreover, let $\varphi : G \to F$ be the $S$-module homomorphism of free $S$-modules given by the matrix $X$.

Now we give a $(\mathbb{Z}^m \times \mathbb{Z}^n)$-grading to the polynomial ring $S$, by setting $\text{mdeg}(x_{ij}) = (e_i, \varepsilon_j)$ where $e_i$ is the $i$-th canonical basis vector of $\mathbb{Z}^m$ and $\varepsilon_j$ is the $j$-th canonical basis vector of $\mathbb{Z}^n$. 
The chain complex $C(\Delta; \varphi)$ inherits this grading. More precisely, for each $i$, the degree of a basis element $b(\sigma; a)$ of $C_i(\Delta; \varphi)$ with $\sigma = \{j_1, \ldots, j_{m+i}\}$ is set to be $(a + 1, \gamma) \in \mathbb{Z}^m \times \mathbb{Z}^n$, where $\gamma = \varepsilon_{j_1} + \cdots + \varepsilon_{j_{m+i}}$, and $1$ is the vector in $\mathbb{Z}^m$ whose entries are all equal to $1$. 
Let $\Delta$ be a simplicial complex, and let $m$ be a positive integer. Then the following conditions are equivalent:

1. $C(\Delta; \varphi)$ is the linear strand of a finitely generated graded $S$-module with initial degree $m$.

2. $\Delta$ has no minimal nonfaces of cardinality $\geq m + 2$. 
Let $C$ be an $m$-uniform clutter, and let $\mathbb{F}$ be the minimal graded free resolution of $J_C$. Then

$$\mathbb{F}^{\text{lin}} \cong \mathcal{C}(\Delta(C); \varphi).$$
Corollary

Let $C$ be an $m$-uniform clutter. Then

$$
\beta_{i,i+m}(JC) = \binom{m+i-1}{m-1} f_{m+i-1}(\Delta(C)),
$$

for all $i$.

Therefore, the length of the linear strand of $JC$ is equal to

$$
\dim \Delta(C) - m + 1,
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Corollary

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for all $i$.

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$$\dim \Delta(C) - m + 1,$$

and in particular, $\text{projdim } JC \geq \dim \Delta(C) - m + 1$. 
Corollary

Let $C$ be an $m$-uniform clutter. Then

$$
\beta_{i,i+m}(J_C) = \binom{m+i-1}{m-1} f_{m+i-1}(\Delta(C)),
$$

for all $i$.

Therefore, the length of the linear strand of $J_C$ is equal to

$$
\dim \Delta(C) - m + 1,
$$

and in particular, $\text{projdim } J_C \geq \dim \Delta(C) - m + 1$. 
Let $C$ be an $m$-uniform clutter. Then the following conditions are equivalent:

1. $J_C$ has a linear resolution.
2. $J_C$ is linearly presented.
3. $C$ is a complete clutter.


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Thanks for your attention.