# On the toric ideal of a matroid and related combinatorial problems

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- ... and by many other ways (circuits, flats, hyperplanes)

• representable matroid: *E* – a finite subset of a vector space

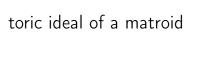
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$$D = (B \setminus e) \cup f$$
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- WEAK: I<sub>M</sub> is generated by quadratic binomials
- CLASSIC: I<sub>M</sub> is generated by quadratic binomials corresponding to symmetric exchanges
- STRONG:  $I_M$  in noncommutative ring  $S_M$  is generated by quadratic binomials corresponding to symmetric exchanges

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- '14 L., Michałek: STRONG for strongly base orderable matroids (contain transversal matroids) and CLASSIC up to saturation for arbitrary matroids

## cyclic ordering conjecture

#### Conjecture (Kajitani, Ueno, Miyano '88)

Let M = (E, r) be a matroid. The following conditions are equivalent:

- for each  $\emptyset \neq A \subset E$  the inequality  $\frac{|A|}{r(A)} \leqslant \frac{|E|}{r(E)}$  holds
- there exists a cyclic ordering it is possible to place elements of E on a circle in such a way that any r(E) cyclically consecutive elements form a basis

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#### Theorem (van den Heuvel, Thomassé '12)

If |E| and r(E) are coprime, then cyclic ordering conjecture holds for M = (E, r).

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  - vertices pairs of bases  $(B_1, B_2)$  which sum to E,
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## Conjecture (cyclic ordering conjecture for a 2-matroid)

There exist complementary bases  $B_1$ ,  $B_2$  in M, such that vertices  $(B_1, B_2)$  and  $(B_2, B_1)$  in the graph  $\mathfrak{B}_2(M)$  are connected by a path of length at most r(E).

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#### Conjecture (Farber, Richter, Shank '85)

For every 2-matroid M the graph  $\mathfrak{B}_2(M)$  is connected.

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- WEAK  $\iff$  for  $k \ge 3$  graph  $\mathfrak{B}_k$  is connected
- STRONG  $\iff$  for  $k \geqslant 2$  graph  $\mathfrak{B}_k$  is connected

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#### Conjecture

Graph  $\mathfrak{B}_2(M)$  has diameter r(E).



 $\Downarrow$ 

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#### Conjecture

Some vertices  $(B_1, B_2), (B_2, B_1)$  are connected. CLASSIC  $\Leftrightarrow$  STRONG

# Thank you!