Representation of Sparse Legendre Expansions

Thomas Peter

University of Göttingen, Institute for Numerical and Applied Mathematics, Lotzestr. 16-18, 37083 Göttingen, Germany.

Gerlind Plonka

University of Göttingen, Institute for Numerical and Applied Mathematics, Lotzestr. 16-18, 37083 Göttingen, Germany.

Daniela Roşca

Technical University of Cluj-Napoca, Department of Mathematics, str. Memorandumului 28, RO-400114 Cluj-Napoca, Romania.

Abstract

We derive a new deterministic algorithm for the computation of a sparse Legendre expansion f of degree N with $M \ll N$ nonzero terms from only $2M$ function resp. derivative values $f^{(j)}(1)$, $j = 0, \ldots, 2M-1$ of this expansion. For this purpose we apply a special annihilating filter method that allows us to separate the computation of the indices of the active Legendre basis polynomials and the evaluation of the corresponding coefficients.

Key words: Legendre polynomials, sparse polynomial expansions, annihilating filters, nonlinear approximation.

1. Introduction

Within the last years, there has been an increasing interest in exploiting sparsity of solutions in suitable bases or frames. Usually, the central issue is the recovery of sparse signals from a rather small set of determining points. Particularly, compressive sensing

Email addresses: `t.peter@math.uni-goettingen.de` (Thomas Peter), `plonka@math.uni-goettingen.de` (Gerlind Plonka), `Daniela.Rosca@math.utcluj.ro` (Daniela Roşca).

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has triggered significant research activity. For example, a trigonometric polynomial of degree N with only $s \ll N$ active terms has been shown to be recovered by $\mathcal{O}(s \log^4(N))$ sampling points that are randomly chosen from a discrete set $\{j/N\}_{j=0}^{N-1}$, (Candès et. al, 2006), or from the uniform measure on $[0, 1]$, (Rauhut, 2007). In (Rauhut & Ward, 2012), a method for recovering of a sparse Legendre expansion of order N with $s \ll N$ active terms is introduced, where $\mathcal{O}(s \log^4(N))$ random samples are taken independently according to the Chebyshev probability measure. These results have been further generalized to sparse spherical harmonic expansions, see (Rauhut & Ward, 2011; Burq et. al, 2011). The recovery algorithms in compressed sensing are usually based on a suitable ℓ^1-minimization method, and exact recovery can be ensured only with a certain probability.

In contrast, there exist also deterministic methods for the recovery of sparse trigonometric functions, based on the classical Prony method (Prony, 1795) or the annihilating filter method. This approach even allows to recover the active real frequencies $f_j \in (-\pi, \pi)$ and the complex coefficients $c_j \neq 0$, of a function

$$h(x) = \sum_{j=1}^{M} c_j \, e^{i f_j x}, \qquad x \in \mathbb{R} \tag{1.1}$$

from the equidistant samples $h(k)$, $k = 0, \ldots, 2M-1$, where we assume that M is known a priori. The original Prony method is based on the idea of separating the determination of the unknown frequencies f_j from the determination of the coefficients c_j. In fact, the sparse sum in (1.1) can be regarded as the solution of a linear difference equation. Considering the corresponding "annihilating polynomial" $p(z) = \prod_{j=1}^{M}(z - e^{if_j}) = \sum_{l=0}^{M} p(l) \, z^l$, in a first step one can determine $p(l)$ from given equidistant samples $h(k)$, $k = 0, \ldots, 2M-1$, by solving a linear system with a suitable Hankel matrix. Then the frequencies f_j are obtained from the zeros e^{if_j} of $p(z)$. Afterwards, the coefficients c_j are simply determined by a linear system involving a Vandermonde matrix.

Unfortunately, the classical Prony method is very sensitive to noise, and numerous modifications have been proposed in order to improve its numerical behavior, see e.g. (Roy & Kailath, 1989; Potts & Tasche, 2010, 2011).

The annihilating filter method can be simply transferred to the recovery of sparse multivariate polynomials in monomial basis. We refer to (Zippel, 1979) for a probabilistic approach that is based on the Berlekamp-Massey algorithm (Massey, 1969). In (Ben-Or & Tiwari, 1988) an exact sparse interpolation algorithm for sparse multivariate black-box polynomials was introduced. There have been several attempts to modify the Ben-Or & Tiwari scheme in order to improve its stability and to reduce the computational costs, see e.g. (Kaltofen & Lee, 2003; Giesbrecht et. al, 2009). Another application of annihilating filters can be found e.g. in (Vetterli et. al, 2002; Maracić & Vetterli, 2004; Berent et. al, 2010) for the exact reconstruction of signals with finite rate of innovation, and for computing shapes from moments (Golub et. al, 1999).

However, the above reconstruction ideas based on the annihilating filter method can not easily be transferred to other polynomial bases $(p_k)_{k=0}^{\infty}$, since they require the property

$$p_k \cdot p_l = p_{k+l}$$

(which is of course satisfied for monomials). In (Lakshman & Saunders, 1995), the authors firstly succeeded to achieve reconstruction formulas also for sparse expansions in the Pochhammer basis $(u_n)_{n=0}^{\infty}$ with $u_0(x) = 1$, $u_n(x) = x(x+1)\dots(x+n-1)$ and in the basis of Chebyshev polynomials of first kind, given by $T_0(x) = 1$, $T_1(x) = x$ and $T_n(x) = 2x\,T_{n-1}(x) - T_{n-2}(x)$ for $n \geq 2$. These generalizations are based on very special properties of these two bases, and there is no straightforward method for generalization to other bases of orthogonal polynomials.

In the present paper, we want to consider a different approach to sparse representation of algebraic polynomials. We want to focus particularly on the basis of Legendre polynomials due to its relevance for spherical harmonics expansions on the sphere and as ansatz functions for the solution of partial differential equations.

In contrast to the above interpolation approaches for sparse polynomial reconstruction, we show that e.g. a sparse Legendre expansion of degree N with only $M \ll N$ nonzero terms

$$f(t) = \sum_{j=1}^{M} c_j P_{e_j}(t),$$

where $0 \leq e_1 < e_2 < \ldots < e_M = N$ are elements of $\{0, 1, \ldots, N\}$, is already exactly given by its function and derivative values $f(1), f'(1), \ldots, f^{(2M-1)}(1)$. Note that the usual Vandermonde-type approach for recovering the indices of active polynomial degrees in the monomial basis of $f(t)$ is not available here, since Vandermonde-type methods need $N+1$ function values of $f(t)$ for an exact recovery, whereas the here proposed algorithm uses only $2M \leq N$ derivative values of $f(t)$. Being interested in a usual Taylor expansion of f, we would also need the derivatives of f up to the degree N of f.

We will provide a recovery algorithm for determining the expansion $f(t)$, i.e., the indices of active Legendre polynomials e_j as well as the corresponding coefficients from the values $f^{(k)}(1)$, $k = 0, \ldots, 2M-1$, where M is essentially smaller than N.

2. Properties of Legendre polynomials

In this section, we shortly summarize the definition and some properties of Legendre polynomials that will be useful in the sequel.

The Legendre polynomials are recursively defined by

$$\begin{aligned} P_0(t) &= 1 \\ P_1(t) &= t \\ P_{n+1}(t) &= \left(\frac{2n+1}{n+1}\right) t\,P_n(t) - \left(\frac{n}{n+1}\right) P_{n-1}(t) \qquad n \geq 2, \end{aligned} \tag{2.1}$$

see e.g. (Koepf, 1998), page 2. They can be written by the monomial expansion

$$P_n(t) = \frac{1}{2^n} \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \binom{n}{r} \binom{2n-2r}{n} t^{n-2r}, \tag{2.2}$$

see (Koepf, 1998), page 1. For the derivatives, from (2.2) we easily obtain the recursion

$$P'_{n+1}(t) = (n+1)P_n(t) + tP'_n(t). \tag{2.3}$$

Further differentiation of (2.3) yields

$$P^{(k)}_{n+1}(t) = (n+k)P^{(k-1)}_n(t) + tP^{(k)}_n(t) \tag{2.4}$$

for $k \geq 1$. This can be seen by induction on k from

$$\begin{aligned} P^{(k+1)}_{n+1}(t) &= (n+k)P^{(k)}_n(t) + P^{(k)}_n(t) + t\,P^{(k+1)}_n(t) \\ &= (n+1+k)\,P^{(k)}_n(t) + t\,P^{(k+1)}_n(t) \end{aligned}$$

for $k \geq 0$. We now derive the following three-term recursion for the k-th derivative of P_n.

Lemma 1. *For the derivatives of the Legendre polynomials we have the recursion*

$$(n+1-k)\,P^{(k)}_{n+1}(t) = (2n+1)tP^{(k)}_n(t) - (n+k)\,P^{(k)}_{n-1}(t) \tag{2.5}$$

for $n \geq 0$ *and* $k = 0, \ldots, n+1$.

Proof. We prove (2.5) by induction on k.
For $k = 0$, the recursion coincides with Bonnet's recursion formula (2.1). Assume now that the formula (2.5) is true for some $k \geq 0$.

Differentiation of (2.5) yields

$$(n+1-k)P^{(k+1)}_{n+1}(t) = (2n+1)tP^{(k+1)}_n(t) + (2n+1)P^{(k)}_n(t) - (n+k)P^{(k+1)}_{n-1}(t).$$

Applying (2.4), we substitute $P^{(k)}_n(t)$ by $\frac{1}{(n+1+k)}\left(P^{(k+1)}_{n+1}(t) - t\,P^{(k+1)}_n(t)\right)$ and obtain

$$\begin{aligned} (n+1-k)P^{(k+1)}_{n+1}(t) = (2n+1)tP^{(k+1)}_n(t) + \frac{2n+1}{n+1+k}\left(P^{(k+1)}_{n+1}(t) - tP^{(k+1)}_n(t)\right) \\ -(n+k)P^{(k+1)}_{n-1}(t) \end{aligned}$$

yielding

$$\frac{(n+k)(n-k)}{n+1+k}P^{(k+1)}_{n+1}(t) = (2n+1)\left(1 - \frac{1}{n+1+k}\right)tP^{(k+1)}_n - (n+k)P^{(k+1)}_{n-1}(t).$$

The formula (2.5) is now obtained for $k+1$ by multiplication with $\frac{n+1+k}{n+k}$. $\square$

We are especially interested in a simple representation of the derivatives of Legendre polynomials P_n at the point 1. Obviously, (2.2) yields

$$P^{(k)}_n(1) = \frac{1}{2^n} \sum_{r=0}^{\lfloor (n-k)/2 \rfloor} (-1)^r \binom{n}{r} \binom{2n-2r}{n} \frac{(n-2r)!}{(n-2r-k)!}. \tag{2.6}$$

We can find a simpler representation as follows.

Theorem 2. *For $n \in \mathbb{N}_0$ and $0 \le k \le n$, the derivatives of Legendre polynomials at the point 1 are given by*

$$P_n^{(k)}(1) = \frac{1}{2^k k!} \frac{(n+k)!}{(n-k)!}. \tag{2.7}$$

In particular, $P_n(1) = 1$ for all $n \ge 0$.

Proof. For $k = n$ and $n \in \mathbb{N}_0$, we observe from (2.6) that

$$P_n^{(n)}(1) = \frac{(2n)!}{2^n n!}.$$

Hence, (2.7) is satisfied for $k = n$.

Now, assuming that (2.7) is true for $n \in \mathbb{N}$ and $l = (n-k), \ldots, n$, we need to show that it is also satisfied for $l = n-k-1$. Indeed, from (2.5) and the induction assumption it follows that

$$\begin{aligned}(k+1) P_n^{(n-k-1)}(1) &= (2n-1)\, P_{n-1}^{(n-k-1)}(1) - (2n-k-2)\, P_{n-2}^{(n-k-1)}(1) \\ &= \frac{(2n-1)(2n-k-2)!}{2^{(n-k-1)}(n-k-1)!\, k!} - \frac{(2n-k-2)(2n-k+3)!}{2^{(n-k-1)}(n-k-1)!(k-1)!} \\ &= \frac{1}{2^{(n-k-1)}(n-k-1)!} \, \frac{(2n-k-2)!}{k!} ((2n-1)-k) \\ &= \frac{1}{2^{(n-k-1)}(n-k-1)!} \, \frac{(2n-k-1)!}{k!},\end{aligned}$$

i.e., the assumption holds for $l = n-k-1$. □

3. Representation of sparse Legendre expansions

We consider now the following sparse Legendre expansion with only M terms

$$f(t) = \sum_{j=1}^{M} c_j \, P_{e_j}(t), \tag{3.1}$$

where $0 \le e_1 < e_2 < \ldots < e_M$ are integers with $e_j \in \{0, 1, \ldots, N\}$, and $N \gg M$. We aim to solve the following problem: Given the function and derivative values

$$f(1),\ f'(1), \ldots, f^{(2M-1)}(1),$$

we want to determine the complete function $f(t)$, i.e., the vector $(e_1, e_2, \ldots, e_M)$ of indices of active Legendre polynomials in this expansion, as well as the corresponding coefficient vector $(c_1, c_2, \ldots, c_M)$.

For this purpose, we consider the (unknown) values

$$z_j := \frac{e_j(e_j+1)}{2}, \qquad j = 1, \ldots, M,$$

and observe from (3.1) and (2.7) that

$$
\begin{aligned}
f(1) &= \sum_{j=1}^{M} c_j, \\
f'(1) &= \sum_{j=1}^{M} c_j P'_{e_j}(1) = \sum_{j=1}^{M} c_j \frac{e_j(e_j+1)}{2} = \sum_{j=1}^{M} c_j z_j, \\
f''(1) &= \sum_{j=1}^{M} c_j P''_{e_j}(1) = \sum_{j=1}^{M} c_j \frac{(e_j-1)(e_j+2)}{4} z_j = \sum_{j=1}^{M} c_j \frac{1}{2}(z_j^2 - z_j),
\end{aligned}
$$

etc. Generally, with the ansatz

$$
f^{(k)}(1) = \sum_{j=1}^{M} c_j g_k(z_j), \tag{3.2}
$$

where $g_k(z)$ is a polynomial of degree k, we find the relation

$$
\begin{aligned}
f^{(k+1)}(1) &= \sum_{j=1}^{M} c_j \frac{(e_j+k+1)(e_j-k)}{2(k+1)} g_k(z_j) \\
&= \sum_{j=1}^{M} c_j g_{k+1}(z_j)
\end{aligned}
$$

yielding

$$
g_{k+1}(z) = \frac{2z - k(k+1)}{2(k+1)} g_k(z) = \left(\frac{z}{k+1} - \frac{k}{2} \right) g_k(z).
$$

Hence, we can write the explicit representation of the polynomials $g_k(z)$ for $k \geq 2$ in the form

$$
\begin{aligned}
g_k(z) &= \left(\frac{z}{k} - \frac{(k-1)}{2} \right) \left(\frac{z}{k-1} - \frac{(k-2)}{2} \right) \cdots \left(\frac{z}{2} - \frac{1}{2} \right) \\
&= \frac{1}{k!} \left(z - \binom{k}{2} \right) \left(z - \binom{k-1}{2} \right) \cdots \left(z - \binom{2}{2} \right) z
\end{aligned}
$$

with $g_0(z) = 1$, $g_1(z) = z$, and $g_2(z) = \frac{1}{2}(z-1)z$. Obviously, the polynomials $g_k(z)$, $k = 0, \ldots, N$, form a basis of the space of algebraic polynomials of at most degree N, and there exists a representation

$$
z^l = \sum_{i=0}^{l} \mu_i^l \, g_i(z) \tag{3.3}
$$

for each $l \in \mathbb{N}_0$, with suitably chosen parameters μ_i^l.

For determining the unknown indices $e_1, \ldots, e_M$ in the sparse Legendre expansion (3.1), we consider now the annihilating polynomial

$$\Lambda(z) := \prod_{j=1}^{M} (z - z_j) = \prod_{j=1}^{M} \left(z - \frac{e_j(e_j+1)}{2} \right).$$

Let λ_k, $k = 0, \ldots, M$, be the coefficients of $\Lambda(z)$ in its monomial representation, i.e.,

$$\Lambda(z) = \sum_{k=0}^{M} \lambda_k \, z^k, \quad \text{with} \quad \lambda_M = 1.$$

Then, we observe that for all $m = 0, 1, 2, \ldots$ we have

$$\sum_{k=0}^{M} \lambda_k \sum_{j=1}^{M} c_j \, z_j^{m+k} = \sum_{j=1}^{M} c_j z_j^m \left(\sum_{k=0}^{M} \lambda_k z_j^k \right) = \sum_{j=1}^{M} c_j z_j^m \, \Lambda(z_j) = 0.$$

Denoting

$$b_k := \sum_{j=1}^{M} c_j \, z_j^k, \qquad k = 0, 1, 2, \ldots$$

we thus have

$$\sum_{k=0}^{M} \lambda_k b_{m+k} = 0, \quad \text{i.e.} \quad \sum_{k=0}^{M-1} \lambda_k \, b_{m+k} = -b_{m+M}$$

for $m = 0, 1, 2, \ldots$.

Supposing that we can compute the values b_k for$k = 0, \ldots, 2M-1$, we obtain the linear Hankel system

$$\begin{pmatrix} b_0 & b_1 & b_2 & \ldots & b_{M-1} \\ b_1 & b_2 & b_3 & \ldots & b_M \\ \vdots & \vdots & & & \vdots \\ b_{M-1} & b_M & b_{M+1} & \ldots & b_{2M-2} \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{M-1} \end{pmatrix} = - \begin{pmatrix} b_M \\ b_{M+1} \\ \vdots \\ b_{2M-1} \end{pmatrix}$$

for determining the coefficients $\lambda_0, \ldots, \lambda_M$ of the annihilating polynomial. Having computed $\Lambda(z)$, we can easily find the desired active indices $e_1, \ldots, e_M$ of the Legendre expansion (3.1) by computing the zeros of $\Lambda(z)$. Afterwards, the coefficients c_j, $j = 1, \ldots, M$ are found by solving the linear system

$$f^{(k)}(1) = \sum_{j=1}^{M} c_j P_{e_j}^{(k)}(1),$$

where $P_{e_j}^{(k)}(1)$ can be computed using formula (2.7).

4. Reconstruction scheme for sparse Legendre expansions

Applying the observations from Section 3, we want to derive a first algorithm for the reconstruction of the active indices e_j, $j = 1, \ldots, M$, of the sparse Legendre expansion in (3.1). For this purpose, we first derive a method for computing the values

$$b_k = \sum_{j=1}^{M} c_j\, z_j^k$$

for $k = 0, \ldots, 2M-1$. By means of (3.2) and (3.3) we observe that

$$b_k = \sum_{j=1}^{M} c_j \sum_{i=0}^{k} \mu_i^k\, g_i(z_j) = \sum_{i=0}^{k} \mu_i^k \sum_{j=1}^{M} c_j\, g_i(z_j) = \sum_{i=0}^{k} \mu_i^k\, f^{(i)}(1).$$

Hence, the b_k can be easily computed from the given values $f^{(i)}(1)$ if the coefficients μ_i^k in (3.3) are given.

We recall that the polynomials $g_k(z)$ are recursively defined by

$$g_k(z) = \frac{1}{k}\left(z - \binom{k}{2}\right) g_{k-1}(z) \tag{4.1}$$

for $k \geq 2$ and with $g_0(z) = 1,\ g_1(z) = z$. Let now a_l denote the coefficients of $g_k(z)$ in the expansion

$$g_k(z) = \sum_{\ell=0}^{k} a_\ell^k\, z^\ell.$$

Hence, the recursion (4.1) yields $a_k^k = \frac{1}{k!}$,

$$a_\ell^k = -\frac{k-1}{2} a_\ell^{k-1} + \frac{1}{k} a_{\ell-1}^{k-1} \qquad \text{for} \qquad \ell = 1, \ldots, k-1,$$

and $a_0^k = 0$ for all $k > 0$. Thus, the coefficients a_ℓ^k can be simply computed in a tabular form starting with $a_1^1 = 1$, $a_1^2 = -\frac{1}{2}$, $a_2^2 = \frac{1}{2}$.

Considering now the relation

$$\begin{pmatrix} g_1(z) \\ g_2(z) \\ g_3(z) \\ \vdots \\ g_n(z) \end{pmatrix} = \begin{pmatrix} a_1^1 & 0 & 0 & \ldots & 0 \\ a_1^2 & a_2^2 & 0 & & 0 \\ a_1^3 & a_2^3 & a_3^3 & & \\ \vdots & & & \ddots & \vdots \\ a_1^n & a_2^n & \ldots & a_{n-1}^n & a_n^n \end{pmatrix} \begin{pmatrix} z \\ z^2 \\ z^3 \\ \vdots \\ z^n \end{pmatrix},$$

we only need to invert this triangular coefficient matrix $\mathbf{A} = (a_\ell^k)_{k,\ell=1}^{2M}$ in order to obtain the coefficients μ_ℓ^k of the expansions

$$z^k = \sum_{\ell=0}^{k} \mu_\ell^k g_\ell(z).$$

For the inverse triangular matrix $\mathbf{M} = \mathbf{A}^{-1} = (\mu_\ell^k)_{k,\ell=1}^{2M}$we find the recursion

$$\mu_\ell^k = -\frac{1}{a_\ell^\ell} \sum_{r=\ell+1}^{k} a_\ell^r\, \mu_r^k = -\ell! \sum_{r=\ell+1}^{k} a_\ell^r\, \mu_r^k.$$

Alternatively, we show that the entries μ_ℓ^k of $\mathbf{M}$ satisfy the following three-term recursion.

Theorem 3. *For $n \in \mathbb{Z}$, $n \geq 2$, let $\mathbf{A}_n := (a_\ell^k)_{k,\ell=1}^n$ be the triangular matrix with the entries*

$$a_\ell^k := \begin{cases} 1 & \text{for } \ell = k = 1, \\ \frac{1}{k} a_{\ell-1}^{k-1} + \frac{k-1}{2} a_\ell^{k-1} & \text{for } 1 \leq \ell \leq k,\ k \geq 2, \\ 0 & \text{for } \ell > k, \end{cases}$$

where we assume that $a_0^k := 0$ for $1 \leq k \leq n$. Then the entries μ_ℓ^k of the inverse matrix $\mathbf{M}_n = \mathbf{A}_n^{-1} = (\mu_\ell^k)_{k,\ell=1}^n$ are of the form

$$\mu_\ell^k := \begin{cases} 1 & \text{for } \ell = k = 1, \\ \ell \mu_{\ell-1}^{k-1} + \frac{\ell(\ell+1)}{2} \mu_\ell^{k-1} & \text{for } 1 \leq \ell \leq k,\ k \geq 2, \\ 0 & \text{for } \ell > k, \end{cases} \tag{4.2}$$

where again $\mu_0^k := 0$ for $k \geq 1$. In particular, $\mathbf{A}_n^{-1} \in \mathbb{Z}^{n \times n}$ is again a lower triangular matrix.

Proof. We prove the assertion by induction on n. For $n = 2$,

$$\mathbf{A}_2 = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad \mathbf{M}_2 = \mathbf{A}_2^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix},$$

and the relation (4.2) can easily be checked. Let us now assume that $\mathbf{C}_n := \mathbf{A}_n \mathbf{M}_n = \mathbf{I}_n$ for some $n \geq 2$, where $\mathbf{I}_n$ denotes the identity matrix of size $n \times n$. We consider the product $\mathbf{C}_{n+1} := \mathbf{A}_{n+1} \mathbf{M}_{n+1}$. Using the induction hypothesis, we only need to check the last row of $\mathbf{C}_{n+1} = (c_r^k)_{k,r=1}^{n+1}$ and find for $1 \leq r \leq n+1$,

$$\begin{aligned} c_r^{n+1} &= \sum_{\ell=1}^{n+1} a_\ell^{n+1} \mu_r^\ell \\ &= \sum_{\ell=r}^{n+1} \left(\frac{1}{n+1} a_{\ell-1}^n - \frac{n}{2} a_\ell^n \right) \mu_r^\ell \\ &= \frac{1}{n+1} \sum_{\ell=r}^{n+1} a_{\ell-1}^n \mu_r^\ell - \frac{n}{2} \sum_{\ell=r}^{n+1} a_\ell^n \mu_r^\ell \\ &= \frac{1}{n+1} \sum_{\ell=r}^{n+1} a_{\ell-1}^n \left(r \mu_{r-1}^{\ell-1} + \frac{r(r+1)}{2} \mu_r^{\ell-1} \right) - \frac{n}{2} \delta_{n,r} \\ &= \frac{r}{n+1} \sum_{\ell=r}^{n+1} a_{\ell-1}^n \mu_{r-1}^{\ell-1} + \frac{r(r+1)}{2(n+1)} \delta_{n,r} - \frac{n}{2} \delta_{n,r} = \delta_{n+1,r}, \end{aligned}$$

where we have used that by the induction hypothesis,

$$\sum_{\ell=r}^{n} a_\ell^n \mu_r^\ell = (\mathbf{C}_n)_{n,r} = \delta_{n,r}$$

with the Kronecker symbol $\delta_{n,r}$. □

Summarizing the reconstruction method from Section 3, we now obtain the following algorithm.

Algorithm for reconstruction of sparse Legendre expansions

Input: M, the number of active terms in the Legendre expansion,
$f^{(k)}(1), \quad k = 0, \ldots, 2M-1$.

1. Compute the coefficients μ_ℓ^k for $k = 0, \ldots, 2M-1$ and $\ell = 0, \ldots, k$ by a triangular scheme:
   $\mu_0^0 := 1$,
   For$\ell = 1, \ldots, 2M-1$ set $\mu_\ell^0 := 0$;
   For$k = 1, \ldots, 2M-1$ compute $\mu_0^k := 0$ and $\mu_k^k := k!$;
   for $\ell = 1, \ldots, k-1$ compute $\mu_\ell^k := \ell \mu_{\ell-1}^{k-1} + \frac{\ell(\ell+1)}{2} \mu_\ell^{k-1}$
   end(ℓ)
   end(k)
2. Compute the coefficients
$$b_k := \sum_{i=0}^{k} \mu_i^k f^{(i)}(1)$$
   for $k = 0, \ldots, 2M-1$.
3. Solve the Hankel system
$$\begin{pmatrix} b_0 & b_1 & b_2 & \ldots & b_{M-1} \\ b_1 & b_2 & b_3 & \ldots & b_M \\ \vdots & \vdots & \vdots & & \vdots \\ b_{M-1} & b_M & b_{M+1} & \ldots & b_{2M-2} \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{M-1} \end{pmatrix} = - \begin{pmatrix} b_M \\ b_{M+1} \\ \vdots \\ b_{2M-1} \end{pmatrix}. \tag{4.3}$$
4. Compute the zeros $z_1, \ldots, z_M$ of the polynomial
$$\Lambda(z) = \sum_{j=0}^{M} \lambda_j z^j$$
   using $\lambda_M = 1$ and λ_j, $j = 0, \ldots, M-1$ from step 3, and compute the positive integers e_j from the relation $\frac{e_j(e_j+1)}{2} = z_j$, i.e.
$$e_j = \sqrt{2z_j + \frac{1}{4}} - \frac{1}{2}.$$

5. Compute the coefficients $c_1, \ldots, c_M$ of the sparse Legendre expansion (3.1) by the linear Vandermonde system

$$f^{(k)}(1) = \sum_{j=1}^{M} c_j P_{e_j}^{(k)}(1)$$
$$= \begin{cases} \sum_{j=1}^{M} c_j \frac{(e_j+k)!}{2^k k!(e_j-k)!} & 0 \le k \le e_j \\ 0 & k > e_j \end{cases}$$

with $k = 0, \ldots, 2M-1$.

Output: c_j, e_j for $j = 1, \ldots, M$.

Finally, we show that the linear system in step 3 of the algorithm can be solved uniquely.

Theorem 4. *The Hankel matrix $\mathbf{H} = (b_{k+\ell})_{k,\ell=0}^{M-1}$ given in (4.3), determined by the values b_k, $k = 0, \ldots, 2M-2$, is invertible.*

Proof. The values b_k are defined by

$$b_k = \sum_{j=1}^{M} c_j z_j^k, \quad k = 0, \ldots, 2M-2.$$

Hence $\mathbf{H} \in \mathbb{R}^{M \times M}$ can be factorized as

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ z_1 & z_2 & \ldots & z_M \\ \vdots & \vdots & & \vdots \\ z_1^{M-1} & z_2^{M-1} & \ldots & z_M^{M-1} \end{pmatrix} \begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_M \end{pmatrix} \begin{pmatrix} 1 & z_1 & \ldots & z_1^{M-1} \\ 1 & z_2 & \ldots & z_2^{M-1} \\ \vdots & \vdots & & \vdots \\ 1 & z_M & \ldots & z_M^{M-1} \end{pmatrix}.$$

Since the coefficients $c_1, \ldots, c_M$ have been assumed to be nonzero, and since the values $z_j = \frac{e_j(e_j+1)}{2}$ are pairwise distinct, it follows that rank $\mathbf{H} = M$. Thus, $\mathbf{H}$ is invertible. □

Remark 5. One may also consider the problem of reconstruction of sparse Legendre expansions if the numer M of active terms is not known a priori, but can be bounded by $\tilde{M} > M$. In this case, M can be estimated by the rank of $\mathbf{H}_{\tilde{M}} = (b_{k+\ell})_{k,\ell=0}^{\tilde{M}-1}$, see (Potts & Tasche, 2010) or by a randomization strategy, see (Kaltofen & Lee, 2003).

5. Numerical results

We have implemented the algorithm introduced in Section 4 in MATLAB with data in double-precision. For the first test we consider the polynomial $f_1(t) = \sum_{j=1}^{3} c_j\, P_{e_j}(t)$ of degree 5492 with coefficients c_j and with active indices e_j of Legendre polynomials as given in the following table:

j	e_j	c_j	$\tilde{e}_j$	$\tilde{c}_j$
1	54	2	53.926321165658	2.000000000000163
2	465	-1	465.000238230820	-1.000000000000165
3	5492	-3	5491.999999999998	-3.000000000000000

Hereby $\tilde{e}_j, \tilde{c}_j$, $j = 1, 2, 3$ denote the calculated indices of active Legendre polynomials and the corresponding coefficients, respectively. Note, that the number $2M = 6$ of needed function resp. derivative values of f is much smaller than the polynomial degree $N = 5492$.

In the second example we consider the polynomial $f_2(t) = \sum_{j=1}^{8} c_j \, P_{e_j}(t)$ of degree 62 with 8 active indices e_j and corresponding coefficients c_j as given in the following table:

j	e_j	c_j	$\tilde{e}_j$	$\tilde{c}_j$
1	5	2	4.9999571908200	1.999999999999872
2	27	-1	27.0013765182337	-0.999999999992209
3	31	-3	31.0132655909449	-3.000000000053989
4	32	3	31.9901046967907	3.000000000049416
5	39	5	38.9999848978734	4.999999999996369
6	47	-5	47.0000002509785	-4.999999999999343
7	53	1	52.9999999718226	9.999999999998783
8	62	-0.2	62.0000000000876	-0.199999999999996

Again $\tilde{e}_j, \tilde{c}_j$, $j = 1, \ldots, 8$, denote the calculated indices of active Legendre polynomials and the corresponding coefficients, respectively. Note that $e_j \in \mathbb{N}$ and hence $\tilde{e}_j$ were rounded to integer numbers to improve the results for $\tilde{c}_j$ in step 5 of the algorithm.

Some further remarks on the numerical evaluation are in order.

Remark 6. For generating the needed values of the derivatives of $f(t)$ it is highly recommended to use (2.7) instead of the recursion-formula (2.1). Formula (2.1) is numerically unstable for computing the function value $P_n(1)$ for larger n.

Remark 7. The matrices $\mathbf{A} = (a_k^\ell)_{k,\ell=1}^{2M-1}$ and $\mathbf{M} = \mathbf{A}^{-1} = (\mu_k^\ell)_{k,\ell=1}^{2M-1}$ are data-independent and thus they can be computed beforehand. Here, the three-term recursion for μ_ℓ^k gives more accurate results than an inversion of the matrix $\mathbf{A}$ because there is no cancellation in the computation of μ_ℓ^k, due to the positive coefficients in (4.2). However, unfortunately, the entries in $\mathbf{M}$ are rapidly increasing with the number M of active basis polynomials and cause numerical instabilities especially for the computation of the coefficients c_j, while for the evaluation of e_j we can use the pre-knowledge that $e_j \in \mathbb{Z}$.

Remark 8. Observe that the condition number $\mathrm{cond}(\mathbf{M})$ does not depend on the polynomial degree N but only on the number M of active terms in the Legendre expansion. But since $\mathrm{cond}(\mathbf{M})$ strongly increases with M, the proposed algorithm is highly sensitive to noise. The question of how to improve the numerical stability of the method will be subject of further research.

Remark 9. We are also interested in generalizing the approach to the recovery of sparse expansions of other polynomial bases (as e.g. Jacobi, Laguerre and Hermite) and of spherical harmonics. Another issue for further considerations is the problem of recovering sparse expansions of orthogonal polynomials using only function values instead of derivative values.

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