

# Random Approximation of Convex Bodies: Monotonicity of the Volumes of Random Tetrahedra

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## Abstract

Choose uniform random points  $X_1, \dots, X_n$  in a given convex set and let  $\text{conv}[X_1, \dots, X_n]$  be their convex hull. It is shown that in dimension three the expected volume of this convex hull is in general not monotone with respect to set inclusion. This answers a question by Meckes in the negative.

The given counterexample is formed by uniformly distributed points in the three-dimensional tetrahedron together with a small perturbation of it. As side result we obtain an explicit formula for all even moments of the volume of a random simplex which is the convex hull of three uniform random points in the tetrahedron and the center of one facet.

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## 1 INTRODUCTION

Let  $K \subset \mathbb{R}^d$  be a convex body, i.e., a compact convex set with nonempty interior. Choose random points  $X_1, \dots, X_n$  independently according to the uniform distribution in  $K$ . The convex hull  $\text{conv}[X_1, \dots, X_n]$  of these random points is a random polytope contained in  $K$ . Since for  $n \rightarrow \infty$  we have  $\text{conv}[X_1, \dots, X_n] \rightarrow K$ , this polytope is a random approximation of the convex body  $K$ .

It seems to be immediate that increasing the convex body  $K$  should also increase its random approximation. Thus the question we want to address in this paper is the following: *Is the expected volume of  $\text{conv}[X_1, \dots, X_n]$  a monotone function in the underlying convex body?*

More precisely, assume that  $L, K$  are two  $d$ -dimensional convex bodies. Choose independent uniform random points  $Y_1, \dots, Y_n$  in  $L$  and  $X_1, \dots, X_n$  in  $K$ . Is it true that  $L \subset K$  implies

$$\mathbb{E}|\text{conv}[Y_1, \dots, Y_n]| \leq \mathbb{E}|\text{conv}[X_1, \dots, X_n]|? \quad (1)$$

Here,  $|A|$  denotes the  $d$ -dimensional Lebesgue measure of the  $d$ -dimensional set  $A$ . The starting point for these investigations should be a check for the first nontrivial case  $n = d + 1$  where the convex hull is the random simplex spanned by the random points. In this form, the question was first raised by Meckes [5] in the context of high-dimensional convex geometry.

In dimension one the monotonicity is immediate. It was proved by Rademacher [7] in 2012 that this is also true in dimension two. Our main result solves the three-dimensional case.

**Theorem 1.** *In  $\mathbb{R}^3$  the expected volume of a random polytope is in general **not** monotone under set inclusion. There are three-dimensional convex sets  $L \subset K$  such that*

$$\mathbb{E}|\text{conv}[Y_1, \dots, Y_4]| > \mathbb{E}|\text{conv}[X_1, \dots, X_4]|, \quad (2)$$

if  $Y_1, \dots, Y_4$  are chosen uniformly in  $L$  and  $X_1, \dots, X_4$  in  $K$ .

That the general question (1) cannot be answered in the positive was already shown by Rademacher who, in a groundbreaking paper, gave counterexamples for dimensions  $d \geq 4$  and  $n = d + 1$ . It remains an open problem whether there is a number  $N$ , maybe depending on  $K$  or only on the dimension of the underlying space, such that monotonicity holds for  $n \geq N$ .

For our proof, we need to construct a pair of convex sets leading to a counterexample. A serious drawback of this approach is that one is forced to compute the expected volume of a random simplex which is known to be a notoriously hard problem. In dimension two, tedious but explicit computations from the nineteenth century yielded several explicit results, but starting with dimension three, the problem turns out to be out of reach in general. The only three-dimensional convex sets where the expected volume of a random simplex is known are the ball [6], the cube [11] and the tetrahedron [1]. And in higher dimensions only the ball allows for explicit results. Since numerical computations in dimension three suggest that in the neighbourhood of the cube and the ball the expected volume of a random simplex is monotone, the only potentially tractable counterexample could be the tetrahedron and a set close to it, which also is in accordance with numerical computations by Rademacher[7], and Reichenwallner and Reitzner[8].

Already the determination of the expected volume of a random simplex in a tetrahedron  $T \subset \mathbb{R}^3$  was extremely hard. This question is known as Klee's problem, and after many attempts, erroneous conjectures and numerical estimates, Reitzner and Buchta [1] proved in a long paper that for uniform random points  $X_1, \dots, X_4$  in a tetrahedron of volume one, we have

$$\mathbb{E}|\text{conv}[X_1, \dots, X_4]| = \frac{13}{720} - \frac{\pi^2}{15\,015} = 0.01739\dots \quad (3)$$

It seems to be out of reach to compute this expectation for any other three-dimensional convex set close to  $T$ . Luckily there is a wonderful alternative approach due to Rademacher, using an infinitesimal variation of convex sets, which is stated in the following Lemma.

**Lemma 1** (Rademacher [7]). *For  $d \in \mathbb{N}$ , monotonicity under inclusion of the map*

$$K \mapsto \mathbb{E}|\text{conv}[X_1, \dots, X_{d+1}]|,$$

where  $K$  ranges over all  $d$ -dimensional convex bodies and  $X_i$  are iid uniform points in  $K$ , holds if and only if we have for each convex body  $K \subseteq \mathbb{R}^d$  and for each  $z \in \text{bd } K$  that

$$\mathbb{E}|\text{conv}[X_1, \dots, X_{d+1}]| \leq \mathbb{E}|\text{conv}[X_1, \dots, X_d, z]|.$$

Hence we could get the counterexample for Theorem 1 if we succeed in computing the expectation  $\mathbb{E}|\text{conv}[X_1, \dots, X_3, z]|$  for a particular  $z \in \text{bd } T$ . Because of symmetry, a suitable choice for  $z$  should be the center  $c$  of one of the facets. Yet after several attempts, we observed that computing  $\mathbb{E}|\text{conv}[X_1, \dots, X_3, c]|$  is even more difficult than (3). Although we only would have to integrate over three rather than four points, this does not simplify the computation: the approach in [1] relied on a computation of the expected number of facets rather than that of the volume, which now involves integrations for two distinct cases, over facets formed by 3 random points and those formed by two random points and  $c$  and which turns out to be even more intricate. Nevertheless we will prove the following proposition.

**Proposition 1.** *For a tetrahedron  $T$  of volume one,  $c$  the centroid of a facet of  $T$  and  $X_1, \dots, X_4$  uniform random points in  $T$ , we have that*

$$\mathbb{E}|\text{conv}[X_1, X_2, X_3, c]| < \frac{13}{720} - \frac{\pi^2}{15\,015} = \mathbb{E}|\text{conv}[X_1, \dots, X_4]|.$$

A combination of this result with Rademacher's Lemma 1 yields Theorem 1. The rigorous bound in Proposition 1 is obtained by combining methods from stochastic geometry with results from approximation theory. In the background, first there is a result about the precise approximation of the absolute value function on  $[-\frac{1}{3}, \frac{1}{3}]$  by suitable even polynomials, Lemma 2. To apply this in our context, we use an explicit result for *all* even moments of  $|\text{conv}[X_1, \dots, X_3, c]|$  which — at a first glance maybe surprisingly — is much easier to obtain than just the single first moment.

**Theorem 2.** *Let  $k \in \mathbb{N}$  and choose three uniform random points  $X_1, \dots, X_3$  in a tetrahedron of volume one. Then it holds:*

$$\mathbb{E}|\text{conv}[X_1, \dots, X_3, c]|^{2k} = \frac{8}{3^{2k-3}} \sum_{\sum_{i=1}^3 k_i = 2k} (-1)^{k'} 3^{k''} \binom{2k}{k_1, \dots, k_{18}} \prod_{i=1}^3 \frac{l_i! m_i! n_i!}{(l_i + m_i + n_i + 3)!},$$

where the range of summation and abbreviations are given in (6). The first five even moments are given at the end of Section 2.

We want to mention an application of random convex hulls to statistics. For random points  $X_1, \dots, X_n$  chosen in an interval  $I \subset \mathbb{R}$ , the convex hull  $\text{conv}[X_1, \dots, X_n]$  is the well-known sample range which can also be defined as the interval  $[X_{[1]}, X_{[n]}]$ , where  $X_{[1]} \leq \dots \leq X_{[n]}$  is the the order statistic of the random points and the endpoints  $X_{[1]}, X_{[n]}$  are the extreme points of the random sample. As mentioned above, it is trivial and immediate that the expected length of this sample range is a monotone function in  $I$ . For a generalization of this question to higher dimensions, one needs to generalize the definition of sample range, order statistic and extreme points. Maybe the most natural extension for higher dimensions is to define the sample range to be the convex hull  $\text{conv}[X_1, \dots, X_n]$ , and the extreme points of the sample are those on the boundary of the sample range, i.e., the vertices of  $\text{conv}[X_1, \dots, X_n]$ .

Hence the question in this paper can be restated: *Is the expected volume of the sample range a monotone function in the underlying distribution?* We answer this question in the negative, and a natural precise formulation of the question for general non-uniform measures would be interesting.

This paper is organized in the following way. In Section 2, we give a series representation for even moments of the volume of a random tetrahedron inside a tetrahedron where one point is fixed to be the centroid of a facet, and we use that to find an exact value for the first thirteen even moments. In Section 3, we compute an upper bound for the expected volume of our random tetrahedron, which is a rational affine combination of those even moments. This upper bound suffices to show that the tetrahedron is a counterexample.

As a general reference for results on random polytopes, we refer to the book on Stochastic and Integral Geometry by Schneider and Weil [10]. More recent surveys are due to Hug [4] and Reitzner [9].

## 2 EVEN MOMENTS OF THE VOLUME OF RANDOM SIMPLICES

Let  $T$  be a tetrahedron of volume one and  $c = (x_c, y_c, z_c)$  the centroid of one of its facets. For random points  $X_1, X_2, X_3 \in T$ , we write  $X_i = (x_i, y_i, z_i), i = 1, 2, 3$ . The volume of the simplex with vertices  $X_1, X_2, X_3$  and  $c$  is given by

$$|\text{conv}[X_1, \dots, X_3, c]| = \left| \frac{1}{6} \det \begin{pmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_c & y_c & z_c & 1 \end{pmatrix} \right| = 6^{-1} |D(x_1, \dots, z_c)|,$$

and hence by the absolute value of a polynomial  $D$  of degree precisely three in the coordinates of  $X_1, X_2, X_3$  and  $c$ . We are interested in the even moments of  $|\text{conv}[X_1, \dots, X_3, c]|$ , where we get rid of the absolute value.

$$\mathbb{E}|\text{conv}[X_1, \dots, X_3, c]|^{2k} = 6^{-2k} \int_T \int_T \int_T D(x_1, \dots, z_c)^{2k} d(x_1, y_1, z_1) d(x_2, y_2, z_2) d(x_3, y_3, z_3).$$

Let  $T_o$  be the specific tetrahedron

$$T_o = \{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0, x + y + z \leq 1\},$$

i.e., that with vertices  $(0, 0, 0), (1, 0, 0), (0, 1, 0)$  and  $(0, 0, 1)$ . Note that the volume of  $T_o$  is  $1/6$ . We choose  $c_o = (1/3, 1/3, 0)$ , the centroid of the facet  $\{(x, y, 0) \in \mathbb{R}^3 : x, y \geq 0, x + y \leq 1\}$ .

Since the expectation  $\mathbb{E}|\text{conv}[X_1, \dots, X_3, c]|$  is invariant under volume-preserving affine transformations, we can use as a representative of a tetrahedron of volume one the tetrahedron

$\sqrt[3]{6}T_o$  and the center  $\sqrt[3]{6}c_o$ . We have:

$$\begin{aligned}
 \mathbb{E}|\text{conv}[X_1, \dots, X_3, c]|^{2k} &= 6^{-2k} \int_{\sqrt[3]{6}T_o} \int_{\sqrt[3]{6}T_o} \int_{\sqrt[3]{6}T_o} D(x_1, \dots, z_{\sqrt[3]{6}c_o})^{2k} \\
 &\quad d(x_1, y_1, z_1) d(x_2, y_2, z_2) d(x_3, y_3, z_3) \\
 &= 6^3 \int_{T_o} \int_{T_o} \int_{T_o} D(x_1, \dots, z_{c_o})^{2k} d(x_1, y_1, z_1) d(x_2, y_2, z_2) d(x_3, y_3, z_3). \quad (4)
 \end{aligned}$$

Expanding the determinant, the polynomial  $D$  can be written as

$$\begin{aligned}
 D(x_1, \dots, z_{c_o}) &= \frac{1}{3} \left( x_1 z_2 - x_1 z_3 - x_2 z_1 + x_2 z_3 + x_3 z_1 - x_3 z_2 - y_1 z_2 + y_1 z_3 \right. \\
 &\quad \left. + y_2 z_1 - y_2 z_3 - y_3 z_1 + y_3 z_2 + 3x_1 y_2 z_3 - 3x_1 y_3 z_2 \right. \\
 &\quad \left. - 3x_2 y_1 z_3 + 3x_2 y_3 z_1 + 3x_3 y_1 z_2 - 3x_3 y_2 z_1 \right).
 \end{aligned}$$

By the Multinomial Theorem, and using the multinomial coefficient

$$\binom{2k}{k_1, \dots, k_{18}} = \frac{(2k)!}{k_1! \cdots k_{18}!},$$

the  $(2k)$ -th power of it can be rewritten as

$$\begin{aligned}
 D(x_1, \dots, z_{c_o})^{2k} &= 3^{-2k} \sum_{\sum_1^{18} k_i = 2k} (-1)^{k'} 3^{k''} \binom{2k}{k_1, \dots, k_{18}} (x_1 z_2)^{k_1} (x_1 z_3)^{k_2} (x_2 z_1)^{k_3} \\
 &\quad \times (x_2 z_3)^{k_4} (x_3 z_1)^{k_5} (x_3 z_2)^{k_6} (y_1 z_2)^{k_7} (y_1 z_3)^{k_8} (y_2 z_1)^{k_9} \\
 &\quad \times (y_2 z_3)^{k_{10}} (y_3 z_1)^{k_{11}} (y_3 z_2)^{k_{12}} (x_1 y_2 z_3)^{k_{13}} (x_1 y_3 z_2)^{k_{14}} \\
 &\quad \times (x_2 y_1 z_3)^{k_{15}} (x_2 y_3 z_1)^{k_{16}} (x_3 y_1 z_2)^{k_{17}} (x_3 y_2 z_1)^{k_{18}} \\
 &= 3^{-2k} \sum_{\sum_1^{18} k_i = 2k} (-1)^{k'} 3^{k''} \binom{2k}{k_1, \dots, k_{18}} \prod_{i=1}^3 x_i^{l_i} y_i^{m_i} z_i^{n_i}. \quad (5)
 \end{aligned}$$

Here for abbreviation we use the following notation:

$$\begin{aligned}
 k' &= k_2 + k_3 + k_6 + k_7 + k_{10} + k_{11} + k_{14} + k_{15} + k_{18}, \\
 k'' &= k_{13} + k_{14} + k_{15} + k_{16} + k_{17} + k_{18}, \\
 l_1 &= k_1 + k_2 + k_{13} + k_{14}, \\
 m_1 &= k_7 + k_8 + k_{15} + k_{17}, \\
 n_1 &= k_3 + k_5 + k_9 + k_{11} + k_{16} + k_{18}, \\
 l_2 &= k_3 + k_4 + k_{15} + k_{16}, \\
 m_2 &= k_9 + k_{10} + k_{13} + k_{18}, \\
 n_2 &= k_1 + k_6 + k_7 + k_{12} + k_{14} + k_{17}, \\
 l_3 &= k_5 + k_6 + k_{17} + k_{18}, \\
 m_3 &= k_{11} + k_{12} + k_{14} + k_{16}, \\
 n_3 &= k_2 + k_4 + k_8 + k_{10} + k_{13} + k_{15}.
 \end{aligned} \tag{6}$$

Integration of the monomials over the tetrahedron  $T_o$  gives

$$\int_{T_o} x^{l_i} y^{m_i} z^{n_i} d(x, y, z) = \underbrace{\int_0^1 \int_0^{1-t} \int_0^{1-t-s} x^{l_i} y^{m_i} z^{n_i} dx dy dz}_{x+y+z \leq 1}$$

and the substitution  $z = t$ ,  $y = s(1 - t)$ ,  $x = r(1 - s)(1 - t)$  yields

$$\begin{aligned}
 &= \int_0^1 r^{l_i} dr \int_0^1 s^{m_i} (1 - s)^{l_i+1} ds \int_0^1 t^{n_i} (1 - t)^{l_i+m_i+2} dt \\
 &= \frac{1}{l_i + 1} B(m_i + 1, l_i + 2) B(n_i + 1, l_i + m_i + 3) \\
 &= \frac{l_i! m_i! n_i!}{(l_i + m_i + n_i + 3)!},
 \end{aligned}$$

where  $B(\cdot, \cdot)$  denotes the Beta function. Combining this with equations (4) and (5) gives

$$\begin{aligned}
 \mathbb{E}|\text{conv}[X_1, \dots, X_3, c]|^{2k} &= \\
 &= \frac{8}{3^{2k-3}} \sum_{\sum_1^{18} k_i = 2k} (-1)^{k'} 3^{k''} \binom{2k}{k_1, \dots, k_{18}} \prod_{i=1}^3 \frac{l_i! m_i! n_i!}{(l_i + m_i + n_i + 3)!},
 \end{aligned}$$

which is Theorem 2. We list the first five even moments of the volume of a random simplex

in a tetrahedron  $T$  of volume one:

$$\begin{aligned}\mathbb{E}|\text{conv}[X_1, \dots, X_3, c]|^2 &= \frac{1}{2000} = 0.0005, \\ \mathbb{E}|\text{conv}[X_1, \dots, X_3, c]|^4 &= \frac{43}{27\,783\,000} \approx 1.54771 \cdot 10^{-6}, \\ \mathbb{E}|\text{conv}[X_1, \dots, X_3, c]|^6 &= \frac{347}{28\,805\,414\,400} \approx 1.20463 \cdot 10^{-8}, \\ \mathbb{E}|\text{conv}[X_1, \dots, X_3, c]|^8 &= \frac{2\,389}{14\,263\,395\,300\,000} \approx 1.67492 \cdot 10^{-10}, \\ \mathbb{E}|\text{conv}[X_1, \dots, X_3, c]|^{10} &= \frac{310483}{90\,249\,636\,885\,408\,000} \approx 3.44027 \cdot 10^{-12}.\end{aligned}$$

### 3 PROOF OF THEOREM 1

As described in Section 2, the  $(2k)$ -th moment of  $|\text{conv}[X_1, \dots, X_3, c]|$  can be computed, with fast increasing complexity in  $k$ . Hence for some  $n \in \mathbb{N}$ , we want to approximate the absolute value function by an even polynomial,

$$P(x) = \sum_{i=0}^n a_i x^{2i} \quad \text{with } P(x) > |x|, \quad \forall x \in \mathbb{R},$$

or, equivalently,  $P(x) > x$  for all  $x \geq 0$ . Note, that in fact we only need  $P(x) > x$  for all  $x \in [0, \frac{1}{3}]$  since the volume of a tetrahedron in  $T$ , where one vertex is fixed to be the centroid of a facet of  $T$ , is not larger<sup>1</sup> than  $1/3$ .

In contrast to the classical problem of *best approximation* of  $|x|$  by polynomials, we are interested in a one-sided approximation and a certain expected value of the polynomial as objective. We use the following result for polynomial interpolation.

**Lemma 2.** *Let  $m \in \mathbb{N}$ ,  $n = 2m + 1$ , and  $0 < x_0 < \dots < x_m$  be given. Then the system of equations*

$$P(x_j) = x_j \quad \text{and} \quad P'(x_j) = 1 \quad \text{for } j = 0, \dots, m$$

*determines uniquely a polynomial  $P(x) = \sum_{i=0}^n a_i x^{2i}$  with the property  $P(x) \geq |x|$  for all  $x \in \mathbb{R}$ .*

*Proof.* Let  $t_j = x_j^2$  and consider the standard Hermite interpolation problem

$$Q(t_j) = f(t_j) \quad \text{and} \quad Q'(t_j) = f'(t_j) \quad \text{for } j = 0, \dots, m$$

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<sup>1</sup> It is immediately clear that the volume of  $\text{conv}[X_1, \dots, X_3, c]$  is increased if the points  $X_1, \dots, X_3$  are moved within their affine hull on an edge of  $T$ . Further keeping  $X_2, X_3, c$  fixed, the volume is increased by moving  $X_1$  along the edge to one of the vertices of  $T$ . Analogously we increase the volume by moving  $X_2, X_3$  into the vertices of  $T$ . Hence for arbitrary choices of  $X_1, \dots, X_3$  the volume of  $\text{conv}[X_1, \dots, X_3, c]$  is always smaller than the volume of the convex hull of three vertices and  $c$  which equals  $\frac{1}{3}$ .

for the functions  $f(t) = \sqrt{t}$  and  $Q(t) = \sum_{i=0}^n a_i t^i$ . The condition  $P'(x_j) = 1$  is equivalent to  $Q'(t_j) = 1/(2\sqrt{t_j}) = f'(t_j)$ . Then the interpolation error fulfills, for some  $\xi \in [t_0, t_m]$ , the estimate

$$f(t) - Q(t) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^m (t - t_j)^2 < 0,$$

where the first equality is well known and can be found e.g. in [3, eq. (2.60)] and the last inequality follows from  $f^{(n+1)}(t) < 0$  for all  $t$ .  $\square$

We note in passing that for even  $n = 2m$  and  $0 < x_0 < \dots < x_m = 1/3$ , we only require the simple interpolation condition  $P(x_m) = x_m$  in the last point and get  $P(x) \geq |x|$  for all  $x \in [-x_m, x_m]$ .

Our aim is to approximate  $\mathbb{E}|\text{conv}[X_1, \dots, X_3, c]|$  from above by an even polynomial  $P$  of degree  $2n$ ,

$$\mathbb{E}|\text{conv}[X_1, \dots, X_3, c]| \leq \mathbb{E}P(|\text{conv}[X_1, \dots, X_3, c]|),$$

which holds if  $|x| \leq P(x)$  on  $[-\frac{1}{3}, \frac{1}{3}]$ . Moreover, the best polynomial for fixed  $n \in \mathbb{N}$  can be found via the linear optimization problem

$$\begin{aligned} \min_P \mathbb{E}P(|\text{conv}[X_1, \dots, X_3, c]|) &= \min_{a_i} \sum_{i=0}^n a_i \mathbb{E}P(|\text{conv}[X_1, \dots, X_3, c]|)^{2i} \\ &\text{s.t. } P(x) \geq x, x \in \left[0, \frac{1}{3}\right]. \end{aligned}$$

Please note that the constraint is infinite dimensional. Relaxing the constraint, we get a lower bound on  $\mathbb{E}P(|\text{conv}[X_1, \dots, X_3, c]|)$  via the finite dimensional linear program

$$\min_P \mathbb{E}P(|\text{conv}[X_1, \dots, X_3, c]|) \quad \text{s.t. } P(x_\ell) \geq x_\ell, x_\ell \in \left[0, \frac{1}{3}\right], \ell = 0, \dots, L.$$

For  $n = 12$  and  $L = 100$  equidistant points  $x_\ell \in [0, 1/3]$ , we numerically compute via Matlab and the optimization toolbox CVX [2]

$$\mathbb{E}P(|\text{conv}[X_1, \dots, X_3, c]|) > 0.01746,$$

yielding that we do not get a sufficiently precise estimate using only  $n = 12$  even moments.

For  $n = 13$  and  $L = 1\,000$ , we solved the above linear program, computed the interpolation nodes with the absolute value function numerically, and rationalized these points to

$$\{x_j : j = 0, \dots, 6 = m\} = \left\{ \frac{1}{83}, \frac{1}{22}, \frac{1}{11}, \frac{2}{15}, \frac{2}{11}, \frac{5}{22}, \frac{4}{15} \right\}.$$

Using these points for the interpolation problem in Lemma 2 gives an even polynomial  $P_{\text{cert}}(x) = \sum_{i=0}^{13} a_i x^{2i}$  of degree 26 with the property  $|x| \leq P_{\text{cert}}(x)$  and explicitly given rational coefficients  $a_0, \dots, a_{13}$ , which can be computed via Mathematica/WolframAlpha by



`CoefficientList[InterpolatingPolynomial[{{(1/83)^2, 1/83, 83/2}, {(1/22)^2, 1/22, 22/2},`  
`{(1/11)^2, 1/11, 11/2}, {(2/15)^2, 2/15, 15/4}, {(2/11)^2, 2/11, 11/4}, {(5/22)^2, 5/22, 22/10},`  
`{(4/15)^2, 4/15, 15/8}}, t], t]].`

Finally, we use the even moments computed in Section 2 to complete the proof of Theorem 1:

$$\begin{aligned} \mathbb{E}|\text{conv}[X_1, \dots, X_3, c]| &\leq \mathbb{E}P_{\text{cert}}(|\text{conv}[X_1, \dots, X_3, c]|) \\ &= \sum_{i=0}^{13} a_i \mathbb{E}|\text{conv}[X_1, \dots, X_3, c]|^{2i} \\ &= \underbrace{921571629635484412042963800745536952467367872279381650093007745647304556888704811540672891617753958471466587267976070649083068559759815228565946584826727651530274961059565653153253525024497186724789134747165293757323427650333181847590012204163284846690767494412592136803582459020919668764372661702456688640600000000}_{=0.0173791\dots} \\ &< \underbrace{\frac{13}{720} - \frac{\pi^2}{15015}}_{=0.01739\dots} = \mathbb{E}|\text{conv}[X_1, \dots, X_4]|. \end{aligned}$$

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## A COEFFICIENTS OF THE CERTIFYING POLYNOMIAL

As outlined above, the coefficients of the certifying polynomial  $P_{\text{cert}}(x) = \sum_{i=0}^{13} a_i x^{2i}$  can be computed via Mathematica/WolframAlpha by

```
CoefficientList[InterpolatingPolynomial[{{(1/83)^2, 1/83, 83/2}, {(1/22)^2, 1/22, 22/2},
{(1/11)^2, 1/11, 11/2}, {(2/15)^2, 2/15, 15/4}, {(2/11)^2, 2/11, 11/4}, {(5/22)^2, 5/22, 22/10},
{(4/15)^2, 4/15, 15/8}}, t], t]].
```

The outcome being the following

$$\begin{aligned}
 a_0 &= \frac{273336906365397426078864562906838565074223151518907516027760030288}{50943620940674447995025845049145113414460531981461799361349568433785}, \\
 a_1 &= \frac{172837051889011535335292359292442626554610051119055059052477233305696451}{3396241396044963199668389669943007560964035465430786624089971228919000}, \\
 a_2 &= \frac{6927588217235178769557406710109274630390932645803307163220690912724619069771}{194070936916855039981050838282457574912230598024616378519426927366800000}, \\
 a_3 &= \frac{1539701310973043540733663073336277968431072883663044622071160541866514737064773}{95333091818805984552095148629979159606008013064723835062174630987200000}, \\
 a_4 &= \frac{88327475345398476059138077417627600090239600279861230859750496031195814132503809067}{21735944934687764477877693887635248390169826978757034394175815865081600000}, \\
 a_5 &= \frac{1506270190983537944333073056260955986674690372081669691029348049238330196984938510981}{2484107992535744511757450730015456958876551654715089645048664670295040000}, \\
 a_6 &= \frac{246633441732207080900477381555665867092061311488114828666516763515718318471689150031929}{4347188986937552895575538777527049678033965395751406878835163173016320000}, \\
 a_7 &= \frac{150101682160042945470190901995986897880722034759499877158323382538946014306167743194748781}{4347188986937552895575538777527049678033965395751406878835163173016320000}, \\
 a_8 &= \frac{216421540830090047619172672364557230894581364702757249314559035543600303694958394153607999}{1552567495334840319848406706259660599297844784196931028155415418934400000}, \\
 a_9 &= \frac{514275187632999910656779297199985716026363828150333582700145623534658255442221093716426217999}{13727965221908061775501701402716998983265153881320232248953146862156800000}, \\
 a_{10} &= \frac{956417972117281671478581649589053868970984424939599287708100675016783608647450833493258197}{14490629956458509651918462591756832260113217985838022929450543910054400}, \\
 a_{11} &= \frac{5596009572109106658454688082500249146975210969609085016785660326167107079744423954987539}{7666999976962174419004477561776101724927628563935461867434150216960}, \\
 a_{12} &= \frac{2049601492698075453523500707185442228654433119966392837533678044726020212011075186145325}{447241665322793507775261191103605933954111666229568608933658762656}, \\
 a_{13} &= \frac{618003329365426042046464616484669501570327832872989878011229369057953701120253938097625}{49693518369199278641695687900400659328234629581063178770406529184}.
 \end{aligned}$$