Nonequispaced fast Fourier transforms without oversampling

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Recently, the fast Fourier transform (FFT) has been generalised for arbitrary sampling nodes by the use of approximation schemes. We show that such nonequispaced FFTs can be implemented without oversampling, i.e., no extra memory besides the input and output array is needed.

1 Introduction

Without doubt, the fast Fourier transform [1] belongs to the algorithms with the greatest influence on the development and practice of science and engineering. It has become of great importance in scientific computing with applications in digital signal and image processing as well as in the numerical solution of differential and integral equations. Two shortcomings of traditional schemes are the need for equispaced sampling and the restriction to the system of complex exponential functions. Both problems have attracted much attention and led to the development of nonequispaced FFTs [2–6]. The common concept in such schemes is to trade exactness for efficiency; instead of precise computations up to machine precision, the proposed methods guarantee a given target accuracy.

State of the art approaches for nonequispaced FFTs rely internally on an oversampled FFT and a dedicated approximation scheme. If the input and output arrays just fit into the main memory, these methods are no longer applicable. Without oversampling, the accuracy of these methods cannot be controlled and numerical experiments indeed show that the approximation fails. We generalise the approach [7] and give error bounds for the case of no oversampling and even undersampling. The number of floating point operations of the new method is within logarithmic terms slightly worse than for the aforementioned algorithms.

2 Local Taylor series expansions

Let an even bandwidth $N \in \mathbb{N}$, a vector of Fourier coefficients $\hat{\mathbf{f}} \in \mathbb{C}^N$, and the trigonometric polynomial $f : [-\frac{1}{2}, \frac{1}{2}) \to \mathbb{C}$,

$$f(x) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{f}_k \mathrm{e}^{-2\pi \mathrm{i} kx}$$

be given. Moreover, let $M \in \mathbb{N}$ and a set of sampling nodes $x_j \in [-\frac{1}{2}, \frac{1}{2})$, $j = 0, \dots, M-1$, be given. The nonequispaced discrete Fourier transform is defined as the evaluation of the trigonometric polynomial f at the nodes x_j. We collect these samples in the vector $\mathbf{f} \in \mathbb{C}^M$, $f_j = f(x_j)$, $j = 0, \dots, M-1$, and denote the nonequispaced Fourier matrix by $\mathbf{A} \in \mathbb{C}^{M \times N}$, $a_{j,k} = \mathrm{e}^{-2\pi \mathrm{i} k x_j}$, $k = -\frac{N}{2}, \dots, \frac{N}{2}-1$, $j = 0, \dots, M-1$. Thus, the nonequispaced discrete Fourier transform is nothing else than the matrix vector product $\mathbf{f} = \mathbf{A}\hat{\mathbf{f}}$, which obviously takes $\mathcal{O}(MN)$ floating point operations. Nonequispaced FFTs [5] reduce this to $\mathcal{O}(N \log N + |\log \varepsilon| M)$, where ε denotes the accuracy of the result. We generalise [7] and approximate f locally by a multivariate or two point Taylor polynomial. A straightforward error analysis yields the following simple results.

Theorem 2.1 *Let f be a trigonometric polynomial of bandwidth $N \in \mathbb{N}$ and $x \in [-\frac{1}{2}, \frac{1}{2})$ be an evaluation node. Moreover, let an under- or oversampling factor $\sigma > 0$ with the corresponding FFT-length $n = \sigma N \in \mathbb{N}$, a cut-off parameter $m \in \mathbb{N}_0$, and the lattice points $y_l = \frac{l}{n}$, $l = -\frac{n}{2}, \dots, \frac{n}{2}$, be given.*

1. *Let f and its first $m-1$ derivatives be evaluated at the nearest lattice point y_l, $|y_l - x| \le |y_r - x|$ for all $r = -\frac{n}{2}, \dots, \frac{n}{2}$. The Taylor expansion about y_l obeys*

$$|f(x) - \sum_{s=0}^{m-1} \frac{f^{(s)}(y_l)}{s!}(x - y_l)^s| \le \frac{\pi^m}{2^m \sigma^m m!} \|f\|_\infty .$$

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2. *Let* $x \in [y_l, y_{l+1}]$*, then the two point Taylor expansion about* y_l *and* y_{l+1}*, i.e.,*

$$p_m(x) := \sum_{s=0}^{m-1} \sum_{t=0}^{m-1-s} \binom{m-1+t}{t} \Big(\frac{(x-y_l)^{s+t}(y_{l+1}-x)^m n^{m+s}}{s!} f^{(s)}(y_l) + \frac{(-1)^t (x-y_{l+1})^{s+t}(x-y_l)^m n^{m+s}}{s!} f^{(s)}(y_{l+1}) \Big)$$

obeys

$$|f(x) - p_m(x)| \le \frac{\pi^{2m}}{2^{2m}\sigma^{2m}(2m)!}\|f\|_\infty.$$

Theorem 2.2 *Now let* f *be a multivariate trigonometric polynomial of multibandwidth* $\mathbf{N} = (N_1, \ldots, N_d) \in \mathbb{N}^d$ *and* $\mathbf{x} \in [-\frac{1}{2}, \frac{1}{2})^2$ *an evaluation node. Moreover, let an under- or oversampling factor* $\sigma > 0$ *with the corresponding FFT-length* $\mathbf{n} = \sigma\mathbf{N} \in \mathbb{N}^d$*, a cut-off parameter* $m \in \mathbb{N}_0$*, and the lattice points* $\mathbf{y_l} = (\frac{l_1}{n_1}, \ldots, \frac{l_d}{n_d})$*;* $l_1 = -\frac{n_1}{2}, \ldots, \frac{n_1}{2}; \ldots; l_d = -\frac{n_d}{2}, \ldots, \frac{n_d}{2}$*; be given. Let* f *and its partial derivatives of order at most* $m-1$ *be evaluated at the nearest lattice point* $\mathbf{y_l}$*. The Taylor expansion about* $\mathbf{y_l}$ *obeys*

$$|f(\mathbf{x}) - \sum_{|\mathbf{s}| \le m-1} \frac{D^{\mathbf{s}} f(\mathbf{y_l})}{\mathbf{s}!}(\mathbf{x}-\mathbf{y_l})^{\mathbf{s}}| \le \frac{d^m \pi^m}{2^m \sigma^m m!}\|f\|_\infty.$$

If $\sigma > 0$ is fixed, the single point Taylor expansion based nonequispaced FFT takes for accuracy $\varepsilon > 0$, for a total number of $N = N_1 \cdot N_2 \cdot \ldots \cdot N_d$ Fourier coefficients, and for M sampling nodes only $\mathcal{O}(|\log \varepsilon|^d (N \log N + M))$ floating point operations. The (univariate) two point Taylor expansion based nonequispaced FFT would take $\mathcal{O}(|\log \varepsilon| N \log N + |\log \varepsilon|^2 M))$ floating point operations, which can be reduced by an f-independent preprocessing step to the above complexity.

The striking point however is the fact that without any oversampling $\sigma = 1$ these algorithms are exponentially accurate with increasing m. This fact is in sharp contrast to the window-based nonequispaced FFTs [2–4].

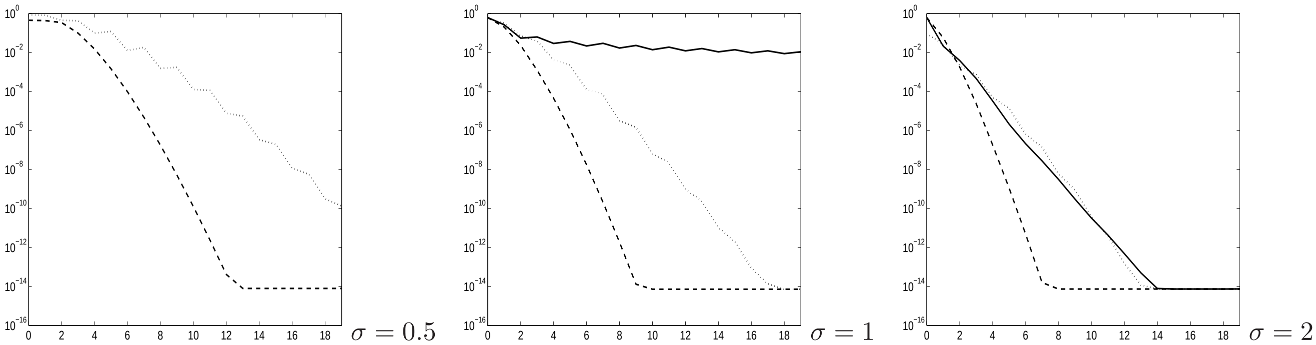

Fig. 1 Error $\max_j |f(x_j) - \tilde{f}_m(x_j)| / \sum_k |\hat{f}_k|$ for increasing cut-off parameter $m = 0, \ldots, 19$ and Gaussian window function (solid), single point Taylor expansion (dotted), and two point Taylor expansion (dashed). The number of Fourier coefficients and the number of nodes are $N = M = 1024$.

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References

[1] M. Frigo and S. G. Johnson, FFTW, C subroutine library, `http://www.fftw.org`.
[2] A. Dutt and V. Rokhlin, SIAM J. Sci. Stat. Comput. **14**, 1368 – 1393 (1993).
[3] G. Beylkin, Appl. Comput. Harmon. Anal. **2**, 363 – 381 (1995).
[4] G. Steidl, Adv. Comput. Math. **9**, 337 – 353 (1998).
[5] D. Potts, G. Steidl, and M. Tasche, Fast Fourier transforms for nonequispaced data: A tutorial, in: Modern Sampling Theory: Mathematics and Applications, edited by J. J. Benedetto and P. J. S. G. Ferreira (Birkhäuser, Boston, 2001), chap. 12, pp. 247 – 270.
[6] J. Keiner, S. Kunis, and D. Potts, NFFT 3.0, C subroutine library, `http://www.tu-chemnitz.de/~potts/nfft`, 2006.
[7] C. Anderson and M. Dahleh, SIAM J. Sci. Comput. **17**, 913 – 919 (1996).