Stable discretizations of the hyperbolic cross fast Fourier transform

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A straightforward discretization of problems in d spatial dimensions with 2^n, $n \in \mathbb{N}$, grid points in each coordinate leads to an exponential growth 2^{dn} in the number of degrees of freedom. Even an efficient algorithm like the d-dimensional fast Fourier transform (FFT) uses $C2^{dn}dn$ floating point operations. This is labeled as the curse of dimensions and the use of sparse grids has become a very popular tool in such situations [2]. We consider Fourier series $f : \mathbb{T}^d \to \mathbb{C}$, $f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \hat{f}_{\boldsymbol{k}} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}$, restrict the frequency domain to the hyperbolic cross

$$H_n^d := \bigcup_{\substack{\boldsymbol{j}\in\mathbb{N}_0^d \\ \|\boldsymbol{j}\|_1=n}} \hat{G}_{\boldsymbol{j}}, \quad \hat{G}_{\boldsymbol{j}} = \times_{l=1}^d \hat{G}_{j_l}, \quad \hat{G}_j = \mathbb{Z} \cap (-2^{j-1}, 2^{j-1}],$$

and ask for the fast approximate evaluation of the d-variate trigonometric polynomial

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in H_n^d} \hat{f}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}, \tag{1}$$

at nodes $\boldsymbol{x}_\ell \in \mathbb{T}^d$, $\ell = 1, \ldots, M$. If we restrict ourselves to sparse grids

$$S_n^d := \bigcup_{\substack{\boldsymbol{j}\in\mathbb{N}_0^d \\ \|\boldsymbol{j}\|_1=n}} G_{\boldsymbol{j}}, \quad G_{\boldsymbol{j}} = \times_{l=1}^d G_{j_l}, \quad G_j = 2^{-j}(\mathbb{Z} \cap [0, 2^j)),$$

we note that the reduced problem size is $|H_n^d| = |S_n^d| = C_d 2^n n^{d-1}$ and a classical result states

Theorem. *[1, 8, 6] For fixed spatial dimension $d \in \mathbb{N}$, the hyperbolic cross FFT, i.e. the computation of* (1) *for all $\boldsymbol{x} \in S_n^d$, takes only $C_d 2^n n^d$ floating point operations.*

This result has been generalized for arbitrary spatial sampling nodes and both algorithms are available in the Matlab toolbox [4].

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Theorem. *[5] For fixed spatial dimension $d \in \mathbb{N}$, the nonequispaced hyperbolic cross FFT, i.e. the computation of* (1) *for all $\boldsymbol{x}_\ell \in \mathbb{T}^d$, $\ell = 1, \ldots, M$, $M = |H_n^d|$, takes only*

$$C_d 2^n n^{2d-2} (|\log \varepsilon| + \log n)^d$$

and assures an error bound

$$\max_{\ell=1,\ldots,M} |f(\boldsymbol{x}_\ell) - \tilde{f}(\boldsymbol{x}_\ell)| \leq \varepsilon \sum_{\boldsymbol{k} \in H_n^d} |\hat{f}_{\boldsymbol{k}}|,$$

where $\tilde{f}(\boldsymbol{x}_\ell)$ denote the computed values.

More recently, we analyzed the numerical stability of these sampling sets and in sharp contrast to the ordinary FFT which is unitary, we found the following negative result.

Theorem. *[9] For fixed spatial dimension $d \in \mathbb{N}$, the evaluation of* (1) *has condition number*

$$c_d 2^{\frac{n}{2}} n^{\frac{2d-3}{2}} \leq \kappa \leq C_d 2^{\frac{n}{2}} n^{2d-2}.$$

More promising, random sampling offers a stable spatial discretization.

Theorem. *[7] For fixed spatial dimension $d \in \mathbb{N}$, fixed $\varepsilon, \delta > 0$, and*

$$M \geq C \varepsilon^{-2} |H_n^d| (\log |H_n^d| + |\log \delta|)$$

independent and uniformly distributed random sampling nodes $\boldsymbol{x}_\ell \in \mathbb{T}^d$, $\ell = 1, \ldots, M$, then with probability at least $1 - \delta$, the evaluation of (1) *has condition number*

$$\kappa \leq \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}.$$

However note that the suggested fast algorithm [5] uses an evaluation on an oversampled sparse grid in its first step and thus suffers from the same instability.

Ongoing work [10] considers lattices as spatial discretization for the hyperbolic cross fast Fourier. These turn out to have quite large cardinality asymptotically but offer perfect stability and outperform known algorithms by at least one order of magnitude with respect to CPU timings for moderate problem sizes.

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