Effective discretization of the two-dimensional wave equation

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The wave equation and associated spherical means are a widespread model in modern imaging modalities like photoacoustic tomography. We consider a discretization for the Cauchy problem of the two dimensional wave equation by plane waves. The considered frequencies lie on a Cartesian or on a polar grid which gives rise to efficient algorithms for the computation of the spherical means. The theoretical findings are illustrated by a some numerical experiments.

1 Introduction

We consider the Cauchy problem for the wave equation

$$\partial_t^2 p(\mathbf{x},t) = \Delta p(\mathbf{x},t), \quad p(\mathbf{x},0) = f(\mathbf{x}),\ \partial_t p(\mathbf{x},0) = g(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2,\ t \in (0,\infty),$$

where the pressure p is sought provided f, g are given. The solution of this equation for sufficiently smooth functions f, g is given by $p(\mathbf{x},t) = \partial_t \left(t \left(\mathcal{N} f\right)(\mathbf{x},t)\right) + t\left(\mathcal{N} g\right)(\mathbf{x},t)$, with the integral operator

$$\mathcal{N} f(\mathbf{y},t) := \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} f\left(\mathbf{y} + t \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}\right) \frac{r\mathrm{d}\varphi \mathrm{d}r}{\sqrt{1-r^2}},$$

see [1, §VI.13, eq. 13,14]. Now let $\mathbf{z} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, $\mathrm{e}_{\mathbf{z}} : \mathbb{R}^2 \to \mathbb{C}$, $\mathrm{e}_{\mathbf{z}}(\mathbf{x}) := \mathrm{e}^{2\pi \mathrm{i} \mathbf{z}\cdot\mathbf{x}}$, $\mathbf{y} \in \mathbb{R}^2$, and $t > 0$ be given. Using [2, Sec. 3.3, (6) and Sec. 12.11, (1)], we have $\mathcal{N}\mathrm{e}_z(\mathbf{y},t) = (2\pi t|\mathbf{z}|)^{-1} \cdot \sin 2\pi t|\mathbf{z}| \cdot \mathrm{e}^{2\pi\mathrm{i}\mathbf{z}\cdot\mathbf{y}}$.

For a discretization parameter $N \in \mathbb{N}$, the aim of this paper is the efficient evaluation of $\mathcal{N} f(\mathbf{y},t)$, $\mathbf{y} \in Y := \{\mathbf{y}_s \in (-\frac{1}{2},\frac{1}{2})^2 : s = 1,\ldots,N\}$, $t \in T := \{t_r \in (0,1) : r = 1,\ldots,N\}$, when f is given by N^2 function values on a grid, or N^2 expansion coefficients in a certain basis. The N spatial nodes Y typically discretize some closed curve around the support of f. Certainly, the use of low-order quadrature formulas is an obvious and well known approach, but typically this leads to $\mathcal{O}\left(N^4\right)$ floating point operations. In the following, we introduce several Fourier based methods with an improved behavior of the running times.

2 Effective discretizations

We define the index set $J := [-\frac{N}{2},\frac{N}{2})^2 \cap \mathbb{Z}^2$ in frequency domain and the sampling grid $X := \{\mathbf{x}_{\mathbf{z}} = \frac{\mathbf{z}}{N} + \frac{1}{2N}, \mathbf{z} \in J\} \subset (-\frac{1}{2},\frac{1}{2})^2$ in space. The discrete Fourier coefficients of a function $f : \mathbb{T}^2 \to \mathbb{R}$ are given by $\hat{f}_{\mathbf{z}} := \frac{1}{N^2}\sum_{\mathbf{x}\in X} f(\mathbf{x})\mathrm{e}^{-2\pi\mathrm{i}\mathbf{x}\cdot\mathbf{z}}$, $\mathbf{z} \in J_N$, and the trigonometric polynomial $p := \sum_{\mathbf{z}\in J_N} \hat{f}_{\mathbf{z}}\mathrm{e}_{\mathbf{z}}$ interpolates the function f on the sampling grid X. Applying the integral operator and evaluating at a center point $\mathbf{y} \in Y$ and a time $t > 0$, yields

$$\mathcal{N} f(\mathbf{y},t) \approx \mathcal{N} p(\mathbf{y},t) = \hat{f}_{\mathbf{0}} + \sum_{\mathbf{z}\in J\setminus\{\mathbf{0}\}} \hat{f}_{\mathbf{z}} \frac{\sin 2\pi t|\mathbf{z}|}{2\pi t|\mathbf{z}|} \mathrm{e}^{2\pi\mathrm{i}\mathbf{z}\cdot\mathbf{y}}, \tag{1}$$

which can be computed by one two-dimensional nonequispaced fast Fourier transform for each time t, see [3] for details. The whole procedure is summarized in Algorithm 1 and has a total complexity of $\mathcal{O}\left(N^3 \log N\right)$.

In our second approach, we evaluate (1) for all times T at once by lifting the two-dimensional problem to a sparse three-dimensional one. Let $\tilde{J} := \{(\mathbf{z},\zeta) \in J \setminus \{\mathbf{0}\} \times \mathbb{R} : |\zeta| = |\mathbf{z}|\} \setminus \{\mathbf{0}\} \subset \mathbb{R}^3$ be the double cone associated to the frequencies $\mathbf{z} \in J$ and their absolute value $|\mathbf{z}|$. Moreover, let the coefficients $\hat{h}_{(\mathbf{z},\zeta)} := \hat{f}_{\mathbf{z}}/\zeta$, $(\mathbf{z},\zeta) \in \tilde{J}$, be defined, then a simple calculation shows

$$\mathcal{N} p(\mathbf{y},t) = \hat{f}_{\mathbf{0}} + \sum_{\mathbf{z}\in J\setminus\{\mathbf{0}\}} \hat{f}_{\mathbf{z}} \frac{\mathrm{e}^{2\pi\mathrm{i}t|\mathbf{z}|} - \mathrm{e}^{-2\pi\mathrm{i}t|\mathbf{z}|}}{4\pi\mathrm{i}t|\mathbf{z}|} \mathrm{e}^{2\pi\mathrm{i}\mathbf{z}\cdot\mathbf{y}} = \hat{f}_{\mathbf{0}} - \frac{\mathrm{i}}{4\pi t} \sum_{(\mathbf{z},\zeta)\in\tilde{J}} \hat{h}_{(\mathbf{z},\zeta)} \mathrm{e}^{2\pi\mathrm{i}(\mathbf{z},\zeta)\cdot(\mathbf{y},t)}. \tag{2}$$

Under the assumption that the center points $\mathbf{y} \in Y$ discretize a smooth curve, an efficient implementation is given by one three dimensional sparse fast Fourier transform [4,5] and has a total complexity of $\mathcal{O}\left(N^2 \log^5 N\right)$ arithmetic operations.

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Algorithm 1 Discrete operator, using nonequispaced fast Fourier transforms

Input

discretization parameter $N \in \mathbb{N}$, samples $\mathbf{f} \in \mathbb{R}^{N^2}$, nodes $\mathbf{y}_s \in (-\frac{1}{2}, \frac{1}{2})^2$, times $t_r > 0$, $s, r = 1, \ldots, N$

Output

$\mathbf{g} \in \mathbb{C}^{N^2}$

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for $\mathbf{z} \in J$ **do**
    $\hat{f}_{\mathbf{z}} := \frac{1}{N^2} \sum_{\mathbf{x} \in X} f(\mathbf{x}) \mathrm{e}^{-2\pi \mathrm{i} \mathbf{z} \cdot \mathbf{x}}$ ▷ FFT
end for
for $r = 1, \ldots, N$ **do**
    for $\mathbf{z} \in J \setminus \{\mathbf{0}\}$ **do**
        $\tilde{h}_{\mathbf{z}}, r := \hat{f}_{\mathbf{z}} \cdot \frac{\sin 2\pi t_r |\mathbf{z}|}{2\pi t_r |\mathbf{z}|}$ ▷ Multiplier
    end for
    $\tilde{h}_{\mathbf{0},r} := \hat{f}_{\mathbf{0}}$
    for $s = 1, \ldots, N$ **do**
        $g_{s,r} := \sum_{\mathbf{z} \in J} \tilde{h}_{\mathbf{z},r} \mathrm{e}^{2\pi \mathrm{i} \mathbf{z} \cdot \mathbf{y}_s}$ ▷ Nonequispaced FFT
    end for
end for

Finally, we restrict to the special case when $Y := \{\mathbf{y}_s := \frac{1}{2}(\cos \frac{2\pi s}{N}, \sin \frac{2\pi s}{N})^\top : s = 0, \ldots, N-1\}$ discretizes a circle and the times $T := \{t_r := \frac{r}{N} : r = 0, \ldots, N-1\}$ are equally spaced. Moreover, the function $f : \mathbb{R}^2 \to \mathbb{C}$ is assumed to be given by the coefficients in the nonharmonic Fourier series $f = \sum_{\mathbf{z} \in J'} \mathrm{e}_{\mathbf{z}}$ with frequencies on the polar grid

$$J' := \left\{ \mathbf{z}_{k,l} := \frac{l}{2} \left(\cos \frac{2\pi k}{N}, \sin \frac{2\pi k}{N} \right)^\top : k, l = 0, \ldots, N-1 \right\}.$$

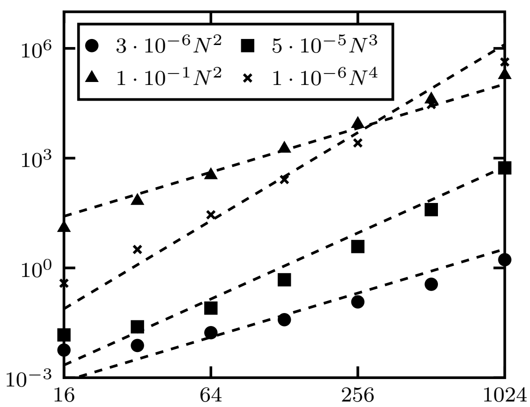
Applying the integral operator, it follows

$$\mathcal{N} f(\mathbf{y}_s, t_r) = \sum_{k=0}^{N-1} \hat{f}_{k,0} + \frac{N}{\pi r} \sum_{l=1}^{N-1} \left[\sum_{k=0}^{N-1} \frac{\hat{f}_{k,l}}{l} \mathrm{e}^{\frac{\pi \mathrm{i} l}{2} \cos \frac{2\pi (k-s)}{N}} \right] \sin \frac{\pi l r}{N}. \tag{3}$$

For each $l = 0, \ldots, N-1$, the inner sum is a cyclic convolution of length N and each outer sum is a discrete sine transform of size N. Thus mapping the coefficients $\hat{\mathbf{f}} \in \mathbb{C}^{N^2}$ to the samples $\mathbf{g} \in \mathbb{C}^{N^2}$ takes $\mathcal{O}(N^2 \log N)$ floating point operations.

3 Numerical experiments

In this section, we consider the running times of the computation of N^2 mean values from N^2 given function samples or expansion coefficients. All numerical experiments were performed in MATLAB R2013a on a computer equipped with a Intel Xeon E7450 CPU with 2.4 GHz and 94 GByte main memory. The figure shows the CPU time in seconds with respect to the discretization parameter N. The applied algorithms are the NFFT based approach (squares), see Equation (1), the sparse FFT based approach (triangles), see Equation (2), the polar grid based approach (circles), see Equation (3), and numerical integration with a simple rectangular rule (crosses).



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References

[1] R. Courant and D. Hilbert, Methods of mathematical physics. Vol. II: Partial differential equations, (Vol. II by R. Courant.) (Interscience Publishers (a division of John Wiley & Sons), New York-London, 1962).
[2] G. N. Watson, A Treatise on the Theory of Bessel Functions (Cambridge University Press, Cambridge, GB, 1966).
[3] J. Keiner, S. Kunis, and D. Potts, ACM Trans. Math. Software **36**(4), Art. 19, 30 (2009).
[4] L. Ying, SIAM J. Sci. Comput. **31**(3), 1678–1694 (2009).
[5] S. Kunis and I. Melzer, SIAM J. Numer. Anal. **50**(3), 1777–1800 (2012).