WORKSHOP ON MOTIVIC AND EQUIVARIANT HOMOTOPY THEORY

UNIVERSITÄT OSNABRÜCK
OCTOBER 4-7, 2017

1. Algebraic geometry lecture series

Markus Spitzweck: Lecture 1. Introduction. For many purposes in algebraic geometry, the Zariski topology on schemes is too coarse. We learn about Grothendieck topologies, in particular the étale site.

Oliver Röndigs: Lecture 2. The Nisnevich topology. The topology mostly used in motivic homotopy theory is the Nisnevich (completely decomposed) topology. We learn the definition, basic properties and where it is more useful than the Zariski or étale topologies, to which it is closely related.

Simon Pepin Lehalleur: Lecture 3. Reductive groups. We learn the definition of reductive linear algebraic groups over a field and how these behave similarly to compact Lie groups.

Holger Brenner: Lecture 4. Algebraic quotients. In algebraic geometry, taking quotients after a group action is not as straightforward as in topology. We learn about these difficulties and the notions of good, categorical or GIT quotients.

2. Minicourse on equivariant homotopy theory, by Irakli Patchkoria

Lecture 1. Equivariant spaces and stabilization. We will start by reviewing the unstable equivariant homotopy theory. We recall basic adjunctions, G-CW complexes, and representation spheres. Next, we will discuss equivariant cohomology theories. The main example here is Bredon cohomology. We will do some elementary calculations of Bredon cohomology groups. Next, we review equivariant Freudenthal suspension theorem and compute stable maps between representation spheres in easy special cases.

Lecture 2. Equivariant spectra and the isotropy separation. This talk will introduce equivariant orthogonal spectra and the genuine equivariant stable homotopy theory. We will briefly explain the role of G-universes and model structures on G-spectra. Next, we construct transfer maps and consider the Mackey structure on equivariant stable homotopy groups. After this we will discuss categorical and geometric fixed point functors and differences between them. The talk will end with the isotropy separation sequence and the Tate square. Having motivic applications in mind, we will in particular concentrate here on the case of the cyclic group of order 2.

Lecture 3. Computations in equivariant stable homotopy theory. We will start by reviewing the Segal–tom Dieck splitting. Then we compute the zeroth equivariant stem. Next we will review the Hill–Hopkins–Ravenel norm and show how to compute the group of components of the norm in the case of the cyclic group of order 2. We will also briefly mention the box product of Mackey functors and the universal coefficient spectral sequence. We will end the talk by explaining how some of these equivariant computations can be used to prove rigidity results for equivariant stable homotopy theories.

3. Minicourse on motivic homotopy theory, by Matthias Wendt

As the title suggests, the goal of the lectures is to provide a tour through some of the basic constructions in motivic homotopy, slightly geared towards the recent applications in algebraic classification problems.

The tour starts with the definition of motivic homotopy, discussion of representability results and ways to compute in the motivic homotopy category. Then we proceed to homotopy modules, a notion which encodes the structures present on cohomology sheaves. At this point we should have enough computational machinery to discuss motivic versions of some of the classical theorems on homotopy of Lie groups, showing in what ways motivic homotopy is different from or similar to classical homotopy. Finally, I'll try to explain how these computations can be translated into some concrete statements about projective modules, octonion algebras or stably trivial quadratic forms.

Lecture 1. Homotopy.

Lecture 2. Cohomology.

Lecture 3. Geometry.

4. Invited talks

Emanuele Dotto: Real topological Hochschild homology and the Hermitian K-theory of $\mathbb{Z}/2$-equivariant ring spectra. Real topological Hochschild homology (THR) is a $\mathbb{Z}/2$-equivariant spectrum introduced by Hesselholt and Madsen as the recipient of a trace map from real algebraic K-theory of discrete rings with anti-involution. In joint work with Moi and Patchkoria we interpret THR as a derived smash product of modules over the Hill–Hopkins–Ravenel norm, and carry out calculations for $\mathbb{F}_p$, group-algebras, and in π_0. In joint work with Ogle we extend the construction of real K-theory to ring spectra, and use the trace to THR to show that the restricted assembly map of the spherical group-ring splits. One can then reformulate the Novikov conjecture in terms of the vanishing of the trace on the kernel of a certain linearization map in rational Hermitian K-theory.

Olivier Haution: Fixed points of p-group actions on projective varieties. I will discuss how the geometry of an algebraic variety restricts the possible actions of finite p-groups on it. I will in particular explain how the computation of numerical invariants can permit to predict the existence of fixed points.

Magdalena Kedziorek: An algebraic model for rational toral G-spectra. The category of G-spectra, for any compact Lie group G is very interesting, but at the same time very complicated. A big part of the interesting information comes from the internally encoded group action while one of the main complications comes from working over the integers. The first step on our understanding is to simplify this category by working over the rationals. This removes the complexity coming from ordinary stable homotopy theory, while leaving much of the information about the group G.

In this talk I will present an algebraic model for a toral part of rational G-spectra for any compact Lie group G. This is a way of understanding rational G-spectra with geometric isotropy in the maximal torus T of G purely in terms of algebraic data. I will discuss the methods behind obtaining algebraic models and what we can deduce from them.

This is joint work with David Barnes and John Greenlees.

5. Contributed talks

Benjamin Böhme: Norm maps and localization of G-spectra. Inverting elements in the homotopy ring of an equivariant ring spectrum might destroy the additional structure of norm maps, as recently studied in work of Blumberg, Hill and Hopkins. We describe this phenomenon explicitly for the G-equivariant sphere spectrum localized at an idempotent and show how it depends on the subgroup structure of G.

Bogdan Gheorghe: Motivic $C(\tau)$-modules and derived BP_*BP-comodules. We will work with motivic spectra over $\operatorname{Spec}\mathbb{C}$. The Tate twist τ living in the cohomology of a point with $\mathbb{F}_p$ coefficients can be lifted to a map between (p-completed) spheres. The cofiber of this map is a 2-cell complex that we denote by $C(\tau)$ with extraordinary properties. A result of Isaksen shows that its homotopy groups are given by the cohomology of the Hopf algebroid (BP_*, BP_*BP), i.e., the E_2 page of the classical Adams–Novikov spectral sequence computing the stable homotopy groups of spheres. We will improve this isomorphism by endowing $C(\tau)$ with an E_∞ ring structure, and promote it to an equivalence of ∞-categories between $C(\tau)$-modules and the algebraic derived category of BP_*BP-comodules. If time permits, we will present some applications of this equivalence of categories.

Christian Wimmer: Rational equivariant commutative ring spectra. Stable homotopy theory simplifies drastically if one considers spectra up to rational equivalence. It is well-known that taking homotopy groups induces an equivalence

$$G - \mathrm{SHC} \simeq_{\mathbb{Q}} \mathrm{gr.} \prod_{(H \leq G)} \mathbb{Q}[WH] - \mathrm{mod}$$

between the genuine G-equivariant stable homotopy category (G finite) and the category of graded modules over the Weyl groups WH indexed by the conjugacy classes of subgroups of G. However, this approach is too primitive to be useful for the comparison of highly structured ring spectra in this setting. I will explain how geometric fixed points equipped with additional norm maps related to the Hill–Hopkins–Ravenel norms can be used to give an algebraic model. They induce an equivalence

$$\mathrm{Com}(G - \mathrm{Sp})_{\mathbb{Q}} \simeq \mathrm{Orb}_G - \mathrm{CDGA}_{\mathbb{Q}}$$

between the homotopy theory of rational commutative G-ring spectra and Orb_G-diagrams in rational commutative differential graded algebras, where Orb_G is the orbit category of the group G. I will also try to indicate the analogous global equivariant statements.