On the condition number of Vandermonde matrices with pairs of nearly-colliding nodes

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We prove upper and lower bounds for the spectral condition number of rectangular Vandermonde matrices with nodes on the complex unit circle. The nodes are "off the grid", pairs of nodes nearly collide, and the studied condition number grows linearly with the inverse separation distance. We provide reasonable sharp constants that are independent from the number of nodes as long as non-colliding nodes are well-separated.

Key words and phrases: Vandermonde matrix, colliding nodes, condition number, frequency analysis, super resolution

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1. Introduction

Vandermonde matrices with complex nodes appear in polynomial interpolation problems and many other fields of mathematics, see e.g. the introduction of [2] and its references. In this paper, we are interested in rectangular Vandermonde matrices with nodes on the complex unit circle and with a large polynomial degree. These matrices generalize the classical discrete Fourier matrices to non-equispaced nodes and the involved polynomial degree is also called bandwidth. The condition number of those matrices has recently become important in the context of stability analysis of super-resolution algorithms like Prony's method [5, 11], the matrix pencil method [10, 15], the ESPRIT algorithm [18, 17], and the MUSIC algorithm [19, 14]. If the nodes of such a Vandermonde matrix are all well-separated, with minimal separation distance greater than the inverse bandwidth, bounds on the condition number are established for example in [4, 12, 15, 2].

If nodes are nearly-colliding, i.e. their distance is smaller than the inverse bandwidth, the behaviour of the condition number is not yet fully understood. The seminal paper [7] coined the term (inverse) super-resolution factor for the product of the bandwidth and the separation distance of the nodes. For M nodes on a grid, the results in [7, 6] imply that the condition number grows like the super-resolution factor raised to the power of $M-1$ if *all* nodes nearly collide. More recently, the practically relevant situation of groups of nearly-colliding nodes was studied in [16, 1, 13, 3]. In different setups and oversimplifying a

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bit, all of these refinements are able to replace the exponent $M-1$ by the smaller number $m-1$, where m denotes the number of nodes that are in the largest group of nearly-colliding nodes. The authors of [16, 1] focus on quite specific quantities in an optimization approach and in the so-called Prony mapping, respectively. In contrast, the condition number or the relevant smallest singular value of Vandermonde matrices with "off the grid" nodes on the unit circle is studied in [13, 3]. While [3] provided the exponent $m-1$ for the first time, the proof technique leads to quite pessimistic constants and more restrictively asks all nodes (including the well-separated ones) to be within a tiny arc of the unit circle. More recently, the second version of [13] provided a quite general framework and reasonable sharp constants, but involves a technical condition which prevents the separation distance from going to zero for a fixed number of nodes and a fixed bandwidth.

Here we present upper and lower bounds for the condition number of Vandermonde matrices with pairs of nearly-colliding nodes. Furthermore, we achieve the expected linear order and all constants are reasonable sharp and absolute. In contrast to the quoted results, the nodes can be placed on the full unit circle and the separation distance is allowed to approach zero. The only technical condition, which seems an artifact of our proof technique, is a logarithmic growth in the separation distance of the well-separated nodes.

The outline of this paper is as follows: Section 2 fixes the notation, recalls results for the case of well-separated nodes, and provides lower bounds for the condition number. In Section 3, we establish upper bounds for nodes that are well-separated from each other except for one pair of nodes that is nearly-colliding. Section 4 goes one step further and looks at the more general case where an arbitrary number of pairs of nodes nearly collide.

2. Preliminaries

Let $\mathbb{T} := \{z \in \mathbb{C}\colon\ |z| = 1\}$ be the complex torus and nodes $\{z_1, \ldots, z_M\} \subset \mathbb{T}$ be parametrized by $z_j = \mathrm{e}^{-2\pi \mathrm{i} t_j}, j = 1 \ldots, M$, such that $t_1 < \cdots < t_M \in [0, 1)$. We fix a degree $n \in \mathbb{N}$ so that $N := 2n + 1 > M$ and set up the rectangular Vandermonde matrix

$$A := \left(z_j^k\right)_{\substack{j=1,\ldots,M\\ |k|\le n}} = \begin{pmatrix} z_1^{-n} & \cdots & z_1^{-1} & 1 & z_1^1 & \cdots & z_1^n \\ \vdots & & \vdots & & \vdots & & \\ z_M^{-n} & \cdots & z_M^{-1} & 1 & z_M^1 & \cdots & z_M^n \end{pmatrix} \in \mathbb{C}^{M\times N}. \tag{2.1}$$

The Dirichlet kernel $D_n\colon \mathbb{R} \to \mathbb{R}$ is given by

$$D_n(t) := \sum_{k=-n}^{n} \mathrm{e}^{2\pi \mathrm{i} k t} = \begin{cases} N, & t \in \mathbb{Z}, \\ \frac{\sin(N\pi t)}{\sin(\pi t)}, & \text{otherwise}, \end{cases} \tag{2.2}$$

so that

$$K := AA^* = \left(D_n\left(t_i - t_j\right)\right)_{\substack{i=1,\ldots,M\\ j=1,\ldots,M}} \in \mathbb{R}^{M\times M}. \tag{2.3}$$

The matrix K is symmetric positive definite and the spectral condition number

$$\operatorname{cond}(A) := \frac{\sigma_{\max}(A)}{\sigma_{\min}(A)} = \sqrt{\|K\|\,\|K^{-1}\|}$$

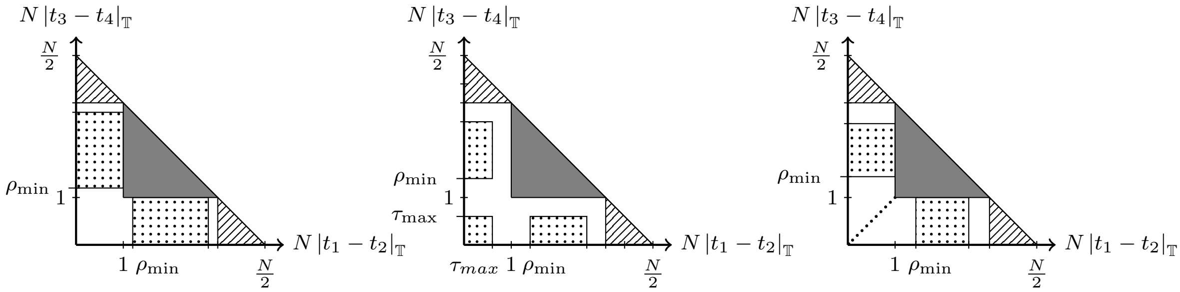
is finite since all nodes are distinct. On the other hand, if two nodes are equal then two rows of A are the same and by continuity the condition number diverges if two nodes collide. The (wrap around) distance of two nodes is given by

$$|t_j - t_\ell|_{\mathbb{T}} := \min_{r\in\mathbb{Z}} |t_j - t_\ell + r| .$$

and we introduce the *normalized separation distance* of the node set as

$$\tau := N \min_{j\neq\ell} |t_j - t_\ell|_{\mathbb{T}} .$$

We call the case $\tau = 1$ *critical separation*, i.e. $\min_{j\neq\ell} |t_j - t_\ell|_{\mathbb{T}} = \frac{1}{N}$, and the cases $\tau \leq 1$ and $\tau > 1$ *nearly-colliding* and *well-separated* respectively. Figure 2.1 illustrates the situation for 4 nodes on the unit circle. The parameter ρ_{min} describes a minimum separation distance of involved non-colliding nodes assumed in the Theorems.



(a) Theorem 3.6 (b) Theorem 4.7 (c) Theorem 4.9

Figure 2.1: Sketch of four-node configurations, $t_1 < t_2 < t_3 < t_4 \in [0, 1)$, $t_1 = 0$, $t_3 = 1/2$, N large enough. dotted: Theorem can be applied, filled: well-separated, lined: 3 nearly-colliding nodes, empty areas: at most 2 nearly-coll. nodes, but not covered by results.

A reasonable result for well-separated nodes is as follows.

Theorem 2.1 ([15, 2])**.** *Let A be a Vandermonde matrix as in* (2.1) *with $\tau > 1$, then*

$$N\left(1 - \frac{1}{\tau}\right) \leq \sigma^2_{\min}(A) \leq N \leq \sigma^2_{\max}(A) \leq N\left(1 + \frac{1}{\tau}\right).$$

In particular we have

$$\operatorname{cond}(A)^2 \leq 1 + \frac{2}{\tau - 1}$$

and for subsequent use we note that $\|K\| \leq N + N/\tau$ and $\|K^{-1}\| = \|A^\dagger\|^2 \leq (N - N/\tau)^{-1}$, where $A^\dagger := A^*(AA^*)^{-1}$ denotes the Moore-Penrose pseudo inverse of A.

Moreover, we have the following lower bound on the condition number. This already shows that the upper bound for well-separated nodes is quite sharp and provides the benchmark for nearly-colliding nodes.

Theorem 2.2 (Lower bound)**.** *Let A be a Vandermonde matrix as in* (2.1)*, then*

$$\sigma^2_{\min}(A) \leq N - |D_n(\tau/N)| \leq N \leq N + |D_n(\tau/N)| \leq \sigma^2_{\max}(A).$$

In particular we have

$$\operatorname{cond}(A)^2 \geq 1 + \frac{2}{\pi\tau - 1}$$

for $\tau \in \mathbb{N} + \frac{1}{2}$, *uniformly in* N *and matching almost the above upper bound.*

For nearly-colliding nodes we have

$$\operatorname{cond}(A)^2 \geq \frac{12}{\pi^2\tau^2} - 1 \geq \frac{1}{\tau^2}$$

for $\tau \leq \sqrt{12/\pi^2 - 1} \approx 0.46$ *and* $\operatorname{cond}(A) \geq \sqrt{6}/\pi\tau \approx 0.77/\tau$ *for all* $\tau \leq 1$.

Proof. Without loss of generality let $t_2 - t_1 = \tau/N$ and consider the upper left 2×2-block in

$$K = \begin{pmatrix} C & * \\ * & * \end{pmatrix}, \quad C := \begin{pmatrix} D_n(0) & D_n(\tau/N) \\ D_n(\tau/N) & D_n(0) \end{pmatrix}.$$

We apply Theorem $A.5$, get

$$\operatorname{cond}(A)^2 = \frac{\lambda_{\max}(K)}{\lambda_{\min}(K)} \geq \frac{\lambda_{\max}(C)}{\lambda_{\min}(C)} = \frac{D_n(0) + |D_n(\tau/N)|}{D_n(0) - |D_n(\tau/N)|} = 1 + \frac{2\,|D_n(\tau/N)|}{N - |D_n(\tau/N)|},$$

and Lemma A.1 yields the assertion. ∎

3. Nodes with one nearly-colliding pair

Definition 3.1. *Let* $M \geq 2$ *and* $0 = t_1 < \cdots < t_M \in [0, 1)$ *such that*

$$|t_1 - t_2|_{\mathbb{T}} = \frac{\tau}{N}, \qquad 0 < \tau \leq 1 \tag{3.1}$$

$$|t_j - t_\ell|_{\mathbb{T}} \geq \frac{\rho}{N},\ j \neq \ell,\ \ell \geq 3, \qquad 1 < \rho < \infty. \tag{3.2}$$

Due to periodicity the choice $t_1 = 0$ and $|t_1 - t_2|_{\mathbb{T}} = \frac{\tau}{N}$ is without loss of generality.

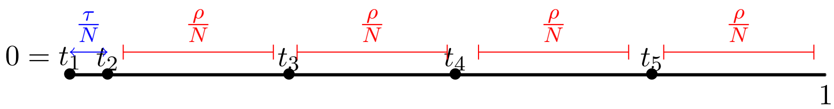


Figure 3.1: Example of a node set with $M = 5$ satisfying Def. 3.1.

Now we estimate an upper bound on the condition number of the hermitian matrix K by bounding $\|K\|$ directly and applying Lemma A.4 to K^{-1} before bounding $\|K^{-1}\|$. For that, we introduce some notation for abbreviation.

Definition 3.2. *We define* $a_1 := \left(z_1^k\right)_{|k| \leq n} \in \mathbb{C}^{1 \times N}$ *and* $A_2 := \left(z_j^k\right)_{\substack{j=2,\ldots,M \\ |k| \leq n}} \in \mathbb{C}^{(M-1) \times N}$ *so that with*

$$a_1 a_1^* = N, \quad K_2 := A_2 A_2^* \quad \text{and} \quad b := A_2 a_1^* = \begin{pmatrix} D_n(\tau/N) \\ D_n(t_3) \\ \vdots \\ D_n(t_M) \end{pmatrix} \tag{3.3}$$

we have the partitioning

$$A = \begin{pmatrix} a_1 \\ A_2 \end{pmatrix} \quad \textit{and} \quad K = \begin{pmatrix} N & b^* \\ b & K_2 \end{pmatrix}, \tag{3.4}$$

where A_2 is a Vandermonde matrix with nodes that are at least $\frac{\rho}{N}$ separated.

Lemma 3.3. *Under the conditions of Definition 3.1 and for $\rho \geq 6$, we have*

$$\|K\| \leq 2.3N.$$

Proof. The key idea is to see the set of nodes as a union of two well-separated subsets and use the existing bounds for these. In contrast to the next chapter, here one of the sets only consist of a single node. We start by noting that Theorem 2.1 and (3.3) yield $\|b\|^2 \leq \|a_1\|^2 \|A_2\|^2 = N\|K_2\|$. Together with the decomposition (3.4), the triangle inequality, Lemma A.6, and Theorem 2.1 we obtain

$$\|K\| \leq \left\| \begin{pmatrix} N & 0 \\ 0 & K_2 \end{pmatrix} \right\| + \left\| \begin{pmatrix} 0 & b^* \\ b & 0 \end{pmatrix} \right\| \leq \|K_2\| + \|b\| \leq N\left(\frac{\rho+1}{\rho} + \sqrt{\frac{\rho+1}{\rho}}\right).$$

■

Lemma 3.4. *Under the conditions of Definition 3.1 and with b as in* (3.3)*, we have*

$$b = \begin{pmatrix} D_n(\tau/N) \\ D_n(t_3) \\ \vdots \\ D_n(t_M) \end{pmatrix} = K_2 \cdot e_1 + r := \begin{pmatrix} D_n(0) \\ D_n(t_3 - \tau/N) \\ \vdots \\ D_n(t_M - \tau/N) \end{pmatrix} + \begin{pmatrix} r_1 \\ \vdots \\ r_{M-1} \end{pmatrix},$$

where $e_1 \in \mathbb{R}^{(M-1)}$ denotes the first unit vector. Additionally, this yields

$$\|r\|^2 \leq (N - D_n(\tau/N))^2 + N^2\tau^2\left(\frac{\pi^4}{12\rho^2} + \frac{1.21\pi}{\rho^3} + \frac{\pi^4}{180\rho^4}\right).$$

Proof. We have $|r_1| = N - D_n(\tau/N)$ and for $j = 2, \ldots, M-1$ the mean value theorem yields

$$|r_j| = |D_n(t_{j+1}) - D_n(t_{j+1} - \tau/N)| = |D_n'(\xi_j)| \frac{\tau}{N}, \quad \xi_j \in (\left|t_{j+1} - \frac{\tau}{N}\right|_{\mathbb{T}}, |t_{j+1}|_{\mathbb{T}}).$$

Note that in the worst case half of the nodes can be as close as possible (under the assumed separation condition) to t_2 not only on its right but also on its left. Hence, for $j = 2, \ldots, \lceil \frac{M}{2} \rceil$, $\xi_j \geq \frac{(j-1)\rho}{N}$ and Lemma A.1 lead to

$$|r_j| \leq N\left(\frac{\pi}{2N|\xi_j|} + \frac{1}{2N^2|\xi_j|^2}\right)\tau \leq N\left(\frac{\pi}{2(j-1)\rho} + \frac{1}{2(j-1)^2\rho^2}\right)\tau.$$

Thus for all nodes we get

$$\sum_{j=2}^{M-1} |r_j|^2 \leq 2\sum_{j=2}^{\lceil M/2 \rceil} |r_j|^2 \leq N^2\tau^2 \left(\frac{\pi^2}{2\rho^2} \underbrace{\sum_{j=1}^{\infty} \frac{1}{j^2}}_{=\frac{\pi^2}{6}} + \frac{\pi}{\rho^3} \underbrace{\sum_{j=1}^{\infty} \frac{1}{j^3}}_{\leq 1.21} + \frac{1}{2\rho^4} \underbrace{\sum_{j=1}^{\infty} \frac{1}{j^4}}_{=\frac{\pi^4}{90}} \right).$$

■

Lemma 3.5. *Under the conditions of Definition 3.1 and for $\rho \geq 5$, we have*

$$\|K^{-1}\| \leq \frac{C(\rho)}{N\tau^2},$$

where

$$C(\rho) = \left(\frac{2\rho-1}{\rho-1} + \sqrt{\frac{\rho}{\rho-1}}\right)\left[2 - \frac{\rho}{\rho-1}\left(1 + \frac{\pi^4}{12\rho^2} + \frac{1.21\pi}{\rho^3} + \frac{\pi^4}{180\rho^4}\right)\right]^{-1}.$$

Proof. We consider K decomposed as in (3.4) and apply Lemma A.4 with respect to K_2 to obtain

$$K^{-1} = \begin{pmatrix} I & 0 \\ -K_2^{-1}b & I \end{pmatrix} \begin{pmatrix} (N - b^* K_2^{-1} b)^{-1} & 0 \\ 0 & K_2^{-1} \end{pmatrix} \begin{pmatrix} I & -b^* K_2^{-1} \\ 0 & I \end{pmatrix}$$

and thus

$$\|K^{-1}\| \leq \left\| \begin{pmatrix} I & 0 \\ -K_2^{-1}b & I \end{pmatrix} \right\|^2 \max\left\{ \|K_2^{-1}\|, \left\|(N - b^* K_2^{-1} b)^{-1}\right\| \right\}.$$

First of all we establish an upper bound for the norm of the triangular matrix. Equation (3.3) and Theorem 2.1 imply

$$\|K_2^{-1}b\| = \|(A_2 A_2^*)^{-1} A_2 a_1^*\| \leq \left\|A_2^\dagger\right\| \|a_1\| \leq \sqrt{\frac{\rho}{\rho-1}}.$$

Together with Lemma A.6 we obtain

$$\left\| \begin{pmatrix} I & 0 \\ -K_2^{-1}b & I \end{pmatrix} \right\|^2 \leq 1 + \|K_2^{-1}b\| + \|K_2^{-1}b\|^2 \leq \frac{2\rho-1}{\rho-1} + \sqrt{\frac{\rho}{\rho-1}}. \tag{3.5}$$

The next step is to bound $(N - b^* K_2^{-1} b)^{-1}$. Lemma 3.4 yields

$$b^* K_2^{-1} b = (K_2 e_1 + r)^* K_2^{-1} (K_2 e_1 + r) = 2D_n(\tau/N) - D_n(0) + r^* K_2^{-1} r.$$

Applying the second part of Lemma 3.4, Lemma A.1, and Theorem 2.1 yields

$$\begin{aligned} N - b^* K_2^{-1} b &\geq 2\left(N - D_n(\tau/N)\right) - \|r\|^2 \|K_2^{-1}\| \\ &\geq \left(N - D_n(\tau/N)\right)\left(2 - \left(N - D_n(\tau/N)\right)\|K_2^{-1}\|\right) - \|K_2^{-1}\| \sum_{j=2}^{M-1} |r_j|^2 \\ &\geq N\tau^2 \left(2 - N\|K_2^{-1}\|\right) - \|K_2^{-1}\| N^2 \tau^2 \left(\frac{\pi^4}{12\rho^2} + \frac{1.21\pi}{\rho^3} + \frac{\pi^4}{180\rho^4}\right) \\ &\geq N\tau^2 \left[2 - \frac{\rho}{\rho-1}\left(1 + \frac{\pi^4}{12\rho^2} + \frac{1.21\pi}{\rho^3} + \frac{\pi^4}{180\rho^4}\right)\right]. \end{aligned}$$

For $\rho \geq 5$, the most inner bracketed term takes values in $(1, 1.4)$ such that the square bracketed term is positive. Forming the reciprocal gives the result, since Theorem 2.1 also implies

$$N\|K_2^{-1}\| \leq \frac{\rho}{\rho-1} \leq \frac{\rho-1}{\rho-2} \leq \left[2 - \frac{\rho}{\rho-1}(1 + \dots)\right]^{-1}. \tag{3.6}$$

∎

Theorem 3.6 (Upper bound)**.** *Under the conditions of Definition 3.1 with $\rho \geq \rho_{\min} = 6$, we have*

$$\operatorname{cond}(A) \leq \frac{4}{\tau}.$$

Proof. The bound follows from Lemmata 3.3 and 3.5 with $C(\rho) \leq C(6) \leq 6.5$. ∎

Remark 3.7. *Lower and upper bounds in Theorems 2.2 and 3.6 yield*

$$\frac{1}{\tau} \leq \operatorname{cond}(A) \leq \frac{4}{\tau}$$

for $\tau \leq 0.46$ and $6 \leq \rho$. The condition on ρ implies that at most $M \leq 1 + N/6 = \mathcal{O}(N)$ nodes can be placed on the unit circle. Some comments regarding what is lost during our proof.

i) The constant in Lemma 3.3 is a numerical value for all $\rho \geq 6$, indeed the proof is valid for all values $\rho > 0$. Moreover, the ρ-dependent inequality contained in the proof is asymptotically sharp in the sense that $\|K\| = 2N$ for $\rho \to \infty$. Lemma 3.4 is asymptotically sharp in the sense that $\|r\| = N - D_n(\tau/N)$ for $\rho \to \infty$.

ii) Finally, the constant $C(\rho)$ in Lemma 3.5 is monotone decreasing in ρ. It is bounded below by 3 which is due to the relatively crude norm estimate on the block triangular factors in the Schur complement decomposition. Note that the left hand side in (3.5) *is bounded from below by $1 + \left\|K_2^{-1}b\right\|^2$. An additional minor improvement on $C(\rho)$ and on the range of admissible values for ρ can be achieved when applying Lemma A.1 to two factors simultaneously.*

4. Pairs of nearly-colliding nodes

In the previous section we analysed the condition number of Vandermonde matrices with nodes of that two nodes are nearly-colliding. Now we study the situation in which the Vandermonde matrix comes from pairs of nearly-colliding nodes.

Definition 4.1. *Let $n \in \mathbb{N}$, $N = 2n+1$, $c \geq 1$ and let $t_1 < \cdots < t_M \subset (0,1]$ for $M \geq 4$ even such that*

$$\begin{aligned}
&\frac{\tau}{N} \leq \left|t_j - t_{j+\frac{M}{2}}\right|_{\mathbb{T}} \leq \frac{c\tau}{N}, \quad j = 1, \ldots, \frac{M}{2}, && 0 < c\tau \leq 1, \\
&\frac{\rho}{N} \leq |t_j - t_\ell|_{\mathbb{T}}, \quad j < \ell, \ell \neq j + \frac{M}{2}, && 1 < \rho < \infty.
\end{aligned}$$

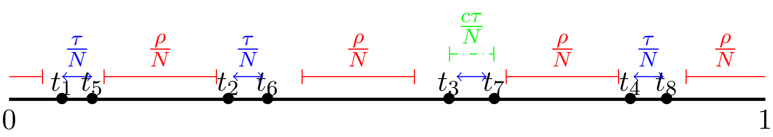


Figure 4.1: Example of a node set with $M = 8$ satisfying Def. 4.1.

For subsequent use, we additionally introduce the following wrap around distance of indices $|j - \ell|' := \min_{r \in \mathbb{Z}} \left|j - \ell + r\frac{M}{2}\right|$ with respect to $\frac{M}{2}$.

Definition 4.2. *We define* $A_1 := \left(z_j^k\right)_{\substack{j=1,\dots,M/2\\|k|\le n}} \in \mathbb{C}^{(M/2)\times N}$ *and* $A_2 := \left(z_j^k\right)_{\substack{j=M/2+1,\dots,M\\|k|\le n}} \in \mathbb{C}^{(M/2)\times N}$ *so that with* $K_1 := A_1 A_1^*$, $K_2 := A_2 A_2^*$ *and* $B := A_2 A_1^*$ *we have the partitioning*

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad K = \begin{pmatrix} K_1 & B^* \\ B & K_2 \end{pmatrix}. \tag{4.1}$$

Note that under the assumptions in Definition 4.1 the Vandermonde matrices A_1 *and* A_2 *are each corresponding to nodes that are at least* ρ/N*-separated.*

The proof technique we use is analogous to the one we used in the case of two nearly-colliding nodes. The difference is that we have a matrix K_1 instead of a scalar and the block B is a matrix instead of a vector. Subsequently, Lemma 4.3 establishes an upper bound on $\|K\|$ and Lemmata 4.4, 4.5, and 4.6 establish an upper bound on $\left\|K^{-1}\right\|$.

Lemma 4.3. *Under the conditions of Definition 4.1, we have*

$$\|K\| \le 2N \cdot \frac{\rho+1}{\rho}.$$

Proof. Similar to Lemma 3.3, we start by noting that $\|B\|^2 \le \|K_1\| \|K_2\|$. Together with the decomposition (4.1), the triangle inequality, Lemma A.6, and Theorem 2.1 this leads to

$$\|K\| \le \left\| \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \right\| + \left\| \begin{pmatrix} 0 & B^* \\ B & 0 \end{pmatrix} \right\| \le \max\left\{\|K_1\|, \|K_2\|\right\} + \sqrt{\|K_1\| \|K_2\|} \le 2N \cdot \frac{\rho+1}{\rho}.$$

■

Lemma 4.4. *Under the conditions of Definition 4.1,* $R_1 := B - K_1$ *fulfils*

$$\|R_1\| \le N - D_n\left(c\tau/N\right) + Nc\tau \left(\frac{\pi(\log\lfloor \frac{M}{4} \rfloor + 1)}{\rho} + \frac{\pi^2}{6\rho^2} \right).$$

Proof. The Dirichlet kernel D_n is monotone decreasing on $[0, 1/N]$. Hence, for the diagonal entries we obtain

$$|(R_1)_{jj}| = \left| D_n\left(t_j - t_{j+\frac{M}{2}}\right) - N \right| = N - D_n\left(t_j - t_{j+\frac{M}{2}}\right) \le N - D_n\left(c\tau/N\right).$$

The off diagonal entries can be bounded by the mean value theorem and Lemma A.1 as

$$|(R_1)_{j\ell}| = \left| D_n\left(t_j - t_\ell\right) - D_n\left(t_{j+\frac{M}{2}} - t_\ell\right) \right| \le \left|D_n'\left(\xi_{j\ell}\right)\right| \frac{c\tau}{N} \le Nc\tau \left(\frac{\pi}{2N\xi_{j\ell}} + \frac{1}{2N^2\xi_{j\ell}^2} \right),$$

where $\left(\left|t_{j+\frac{M}{2}} - t_\ell\right|_{\mathbb{T}}, |t_j - t_\ell|_{\mathbb{T}}\right) \ni \xi_{j\ell} \ge |j-\ell|' \rho/N$ implies

$$|(R_1)_{j\ell}| \le Nc\tau \left(\frac{\pi}{2\rho\,|j-\ell|'} + \frac{1}{2\rho^2(|j-\ell|')^2} \right) =: (\widetilde{R}_1)_{j\ell}$$

for $j, \ell = 1, \dots, \frac{M}{2}$, $j \ne \ell$. Additional, we set $(\widetilde{R}_1)_{jj} := N - D_n\left(c\tau/N\right)$. We bound the spectral norm of R_1 by the one of the real symmetric matrix $\widetilde{R}_1$ using Lemma A.2 and proceed by

$$\|R_1\| \le \left\|\widetilde{R}_1\right\| \le \left\|\widetilde{R}_1\right\|_\infty \le N - D_n\left(c\tau/N\right) + 2Nc\tau \sum_{j=1}^{\lfloor \frac{M}{4} \rfloor} \left(\frac{\pi}{2j\rho} + \frac{1}{2j^2\rho^2} \right),$$

from which the assertion follows. ■

Lemma 4.5. *Under the conditions of Definition 4.1, $R_1 = B - K_1$ and $R_2 := B - K_2$ fulfil*

$$\|2NI + R_1^* + R_2\| \leq 2D_n(\tau/N) + c^2\tau^2 N \left(\frac{\pi^2(\log(\lfloor \frac{M}{4} \rfloor) + 1)}{\rho} + \frac{\pi^3}{3\rho^2} + \frac{2.42}{\rho^3} \right).$$

Proof. First note that

$$(R_1^* + R_2)_{j\ell} = D_n\left(t_{j+\frac{M}{2}} - t_\ell\right) + D_n\left(t_j - t_{\ell+\frac{M}{2}}\right) - D_n\left(t_{j+\frac{M}{2}} - t_{\ell+\frac{M}{2}}\right) - D_n(t_j - t_\ell). \quad (4.2)$$

Monotonicity of the Dirichlet kernel D_n on $t \in [0, 1/N]$ gives

$$|(2NI + R_1^* + R_2)_{jj}| = 2\left|D_n\left(t_{j+\frac{M}{2}} - t_j\right)\right| \leq 2D_n(\tau/N)$$

for $j = \ell$. For each fixed off diagonal entry $j \neq \ell$, the matrix $2NI$ has no contribution. We write the node $t_{j+M/2}$ as a perturbation of t_j by $h_j := t_{j+M/2} - t_j$ and expand the Dirichlet kernel by its Taylor polynomial of degree 2 in the point $\hat{h} := t_j - t_\ell + \frac{h_j - h_\ell}{2}$. Using

$$D_n(h) = D_n(\hat{h}) + D_n(\hat{h})(h - \hat{h}) + \frac{D_n''(\xi)}{2}(h - \hat{h})^2$$

for some $\xi \in [\hat{h}, h] \cup [h, \hat{h}]$ in (4.2), the constant term as well as the linear term cancel out and we get

$$\begin{aligned} &D_n(t_j + h_j - t_\ell) + D_n(t_j - t_\ell - h_\ell) - D_n(t_j + h_j - t_\ell - h_\ell) - D_n(t_j - t_\ell) \\ &= \frac{1}{8}\left(D_n''(\xi_1)(h_j + h_\ell)^2 + D_n''(\xi_2)(h_j + h_\ell)^2 + D_n''(\xi_3)(h_j - h_\ell)^2 + D_n''(\xi_4)(h_j - h_\ell)^2\right). \end{aligned}$$

Lemma A.1 and $\xi_1, \ldots, \xi_4 \geq |j - \ell|' \rho/N$ imply

$$|(R_1^* + R_2)_{j\ell}| \leq \frac{N^3}{4}\left(\frac{\pi^2}{2|j - \ell|'\rho} + \frac{\pi}{(|j - \ell|')^2\rho^2} + \frac{1}{(|j - \ell|')^3\rho^3}\right)\left((h_j + h_\ell)^2 + (h_j - h_\ell)^2\right)$$

and hence by $h_j, h_\ell \leq c\tau/N$

$$|(2NI + R_1^* + R_2)_{j\ell}| \leq Nc^2\tau^2\left(\frac{\pi^2}{2|j - \ell|'\rho} + \frac{\pi}{|j - \ell|'^2\rho^2} + \frac{1}{|j - \ell|'^3\rho^3}\right).$$

The matrix $2NI + R_1^* + R_2$ is real symmetric so that

$$\begin{aligned} \|2NI + R_1^* + R_2\| &\leq \|2NI + R_1^* + R_2\|_\infty \\ &\leq 2D_n(\tau/N) + 2\sum_{j=1}^{\lfloor \frac{M}{4} \rfloor} Nc^2\tau^2\left(\frac{\pi^2}{2j\rho} + \frac{\pi}{j^2\rho^2} + \frac{1}{j^3\rho^3}\right) \\ &\leq 2D_n(\tau/N) + 2c^2\tau^2 N\left(\frac{\pi^2(\log(\lfloor \frac{M}{4} \rfloor) + 1)}{2\rho} + \frac{\pi^3}{6\rho^2} + \frac{1.21}{\rho^3}\right) \end{aligned}$$

and therefore the result holds. ∎

Lemma 4.6. *Under the conditions of Definition 4.1 with $\tau \leq 1/2$ and $\rho \geq 2$, such that*

$$\begin{aligned}\tilde{C}(\tau,\rho,c,M) &:= 2 - \frac{c^2\pi^2(\log(\lfloor \frac{M}{4} \rfloor)+1)}{\rho} - \frac{c^2\pi^3}{3\rho^2} - \frac{2.42c^2}{\rho^3} \\ &\quad - \frac{\rho}{(\rho-1)}\left(\frac{c^2\pi^2}{6}\tau + \frac{c\pi(\log(\lfloor \frac{M}{4} \rfloor)+1)}{\rho} + \frac{c\pi^2}{6\rho^2}\right)^2\end{aligned}$$

is positive, we have

$$\|K^{-1}\| \leq \frac{C(\tau,\rho,c,M)}{N\tau^2},$$

where

$$C(\tau,\rho,c,M) := \left(\frac{2\rho}{\rho-1} + \sqrt{\frac{\rho+1}{\rho-1}}\right)/\tilde{C}(\tau,\rho,c,M).$$

Figure 4.2 visualizes the values of the constant $\tilde{C}(\tau,\rho,c,M)$ with respect to ρ and τ. Please note that 1) increasing the constant c by a factor $\sqrt{2}$ has to be compensated approximately by halving τ and doubling ρ and 2) increasing the number of nodes M from 4 to 64 has to be compensated approximately by tripling ρ.

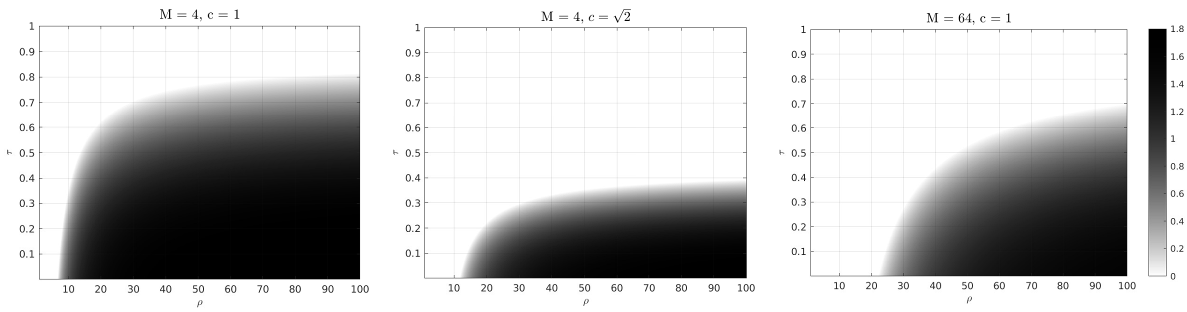


Figure 4.2: Values of $\tilde{C}(\tau,\rho,c,M)$ in Lemma 4.6 depending on τ and ρ for different M and c. Negative values are set to zero.

Proof. We proceed analogously to Lemma 3.5 and apply Lemma A.4 to the matrix K decomposed as in (4.1) and obtain

$$\|K^{-1}\| \leq \max\{\|K_1^{-1}\|, \|(K_2 - BK_1^{-1}B^*)^{-1}\|\} \left\|\begin{pmatrix} I & 0 \\ -BK_1^{-1} & I \end{pmatrix}\right\|^2. \tag{4.3}$$

Definition 4.2 and Theorem 2.1 yield

$$\|BK_1^{-1}\| \leq \|A_2\| \left\|A_1^\dagger\right\| \leq \sqrt{\frac{\rho+1}{\rho-1}},$$

together with Lemma A.6, we obtain

$$\left\|\begin{pmatrix} I & 0 \\ -BK_1^{-1} & I \end{pmatrix}\right\|^2 \leq 1 + \|BK_1^{-1}\| + \|BK_1^{-1}\|^2 \leq \frac{2\rho}{\rho-1} + \sqrt{\frac{\rho+1}{\rho-1}}.$$

Now we estimate $\|(K_2 - BK_1^{-1}B^*)^{-1}\|$, which is done by the following steps.

i) First note that $I - A_1^\dagger A_1$ is an orthogonal projector and thus Theorem 2.1 implies

$$\left\| K_2 - BK_1^{-1}B^* \right\| \leq \|A_2\| \left\| I - A_1^\dagger A_1 \right\| \|A_2^*\| \leq \|A_2\|^2 < 2N.$$

We now apply Lemma A.3 with $\eta = 2N$, use the identities $R_1 = B - K_1$ and $R_2 = B - K_2$, apply the triangular inequality, and the sub-multiplicativity of the matrix norm to get

$$\begin{aligned} \left\| (K_2 - BK_1^{-1}B^*)^{-1} \right\| &\leq \frac{1}{2N - \left\| 2NI - K_2 + BK_1^{-1}B^* \right\|} \\ &\leq \frac{1}{2N - \|2NI + R_1^* + R_2\| - \|R_1\|^2 \left\| K_1^{-1} \right\|}. \end{aligned} \tag{4.4}$$

ii) Lemma 4.5 leads to

$$2N - \|2NI + R_1^* + R_2\| \geq 2(N - D_n(\tau/N)) - c^2\tau^2 N \left(\frac{\pi^2(\log(\lfloor \frac{M}{4} \rfloor) + 1)}{\rho} + \frac{\pi^3}{3\rho^2} + \frac{2.42}{\rho^3} \right).$$

iii) We apply Theorem 2.1 and Lemma 4.4 to get

$$\|R_1\|^2 \left\| K_1^{-1} \right\| \leq \frac{\rho}{N(\rho - 1)} \left[N - D_n(c\tau/N) + Nc\tau \left(\frac{\pi(\log\lfloor \frac{M}{4} \rfloor + 1)}{\rho} + \frac{\pi^2}{6\rho^2} \right) \right]^2.$$

iv) We apply the estimates for the Dirichlet kernel $N - D_n(\tau/N) \geq N\tau^2$ in ii) and $N - D_n(c\tau/N) \leq N\frac{\pi^2}{6}c^2\tau^2$ in iii), see Lemma A.1, and insert this in (4.4) to get finally

$$\begin{aligned} \left\| (K_2 - BK_1^{-1}B^*)^{-1} \right\| \leq \frac{1}{N\tau^2} \Bigg[2 &- \frac{c^2\pi^2(\log(\lfloor \frac{M}{4} \rfloor) + 1)}{\rho} - \frac{c^2\pi^3}{3\rho^2} - \frac{2.42c^2}{\rho^3} \\ &- \frac{\rho}{(\rho - 1)} \left(\frac{c^2\pi^2}{6}\tau + \frac{c\pi(\log(\lfloor \frac{M}{4} \rfloor) + 1)}{\rho} + \frac{c\pi^2}{6\rho^2} \right)^2 \Bigg]^{-1}. \end{aligned}$$

This upper bound also bounds the maximum in (4.3) since for all $\tau \leq 1/2$ and $\rho \geq 2$ together with Theorem 2.1

$$\left\| K_1^{-1} \right\| \leq \frac{2}{N} \leq \frac{1}{2N\tau^2} \leq \frac{1}{N\tau^2}[2 - \cdots]^{-1}.$$

■

Theorem 4.7 (Upper bound)**.** *Under the conditions of Definition 4.1 with* $M \geq 4$*,* $\tau \leq \tau_{max} = \frac{1}{4c^2}$ *and* $\rho \geq \rho_{\min} = 10c^2(\log(\lfloor \frac{M}{4} \rfloor) + 1)$*, we have*

$$\operatorname{cond}(A) \leq \frac{5}{\tau}.$$

Proof. In Lemma 4.6 the constant $C(\tau, \rho, c, M)$ is monotone increasing in τ and monotone decreasing in ρ. Hence, after plugging in the bounds for τ and ρ in our assumptions it is easy to see that the constant $C(\frac{1}{4c^2}, 10c^2(\log(\lfloor \frac{M}{4} \rfloor) + 1), c, M)$ is monotone decreasing in c and M respectively. Therefore, we get $C(\tau, \rho, c, M) \leq C(1/4, 10, 1, 4) \leq 11.3$, so that $\left\| K^{-1} \right\| \leq 11.3N^{-1}\tau^{-2}$. Together with the bound $\|K\| \leq 22N/10 = 2.2N$ from Lemma 4.3, we obtain the result. ■

If each pair of nearly-colliding nodes has the same separation distance, i.e. $c = 1$, we can improve the upper bound in the sense that restrictions on τ except for $\tau \le 1$ can be dropped. In order to obtain the same constant, we have to increase the restrictions on ρ slightly.

Lemma 4.8. *Under the conditions of Definition 4.1 with $c = 1$, such that*

$$\begin{aligned}\tilde{C}(\rho, M) := 2 &- \frac{\pi^2(\log(\lfloor \frac{M}{4} \rfloor) + 1)}{\rho} - \frac{\pi^3}{3\rho^2} - \frac{2.42}{\rho^3} \\ &- \frac{\rho}{\rho - 1} - \frac{2\pi(\log(\lfloor \frac{M}{4} \rfloor) + 1)}{(\rho - 1)} - \frac{\pi^2}{3\rho(\rho - 1)} \\ &- \frac{\pi^2(\log(\lfloor \frac{M}{4} \rfloor) + 1)^2}{\rho(\rho - 1)} - \frac{\pi^3(\log(\lfloor \frac{M}{4} \rfloor) + 1)}{3\rho^2(\rho - 1)} - \frac{\pi^4}{36\rho^3(\rho - 1)}\end{aligned}$$

is positive, we have

$$\|K^{-1}\| \le \frac{C(\rho, M)}{N\tau^2},$$

where $C(\rho, M) := \left(\frac{2\rho}{\rho-1} + \sqrt{\frac{\rho+1}{\rho-1}}\right) / \tilde{C}(\rho, M)$.

Proof. The proof is analogous to that of Lemma 4.6, the only difference is in step iv). Setting $c = 1$ in ii) and iii), expanding the squared bracket in iii) and inserting this into (4.4) leads to

$$\begin{aligned}\left\|(K_2 - BK_1^{-1}B^*)^{-1}\right\| \le \Bigg[& 2\left(N - D_n\left(\tau/N\right)\right) - N\tau^2 \left(\frac{\pi^2(\log(\lfloor \frac{M}{4} \rfloor) + 1)}{\rho} + \frac{\pi^3}{3\rho^2} + \frac{2.42}{\rho^3}\right) \\ &- \frac{\rho}{N(\rho - 1)} \left(N - D_n\left(\tau/N\right)\right)^2 \\ &- \frac{\rho}{\rho - 1} 2\tau \left(N - D_n\left(\tau/N\right)\right) \left(\frac{\pi(\log(\lfloor \frac{M}{4} \rfloor) + 1)}{\rho} + \frac{\pi^2}{6\rho^2}\right) \\ &- N\tau^2 \frac{\rho}{\rho - 1} \left(\frac{\pi^2(\log(\lfloor \frac{M}{4} \rfloor) + 1)^2}{\rho^2} + \frac{\pi^3(\log(\lfloor \frac{M}{4} \rfloor) + 1)}{3\rho^3} + \frac{\pi^4}{36\rho^4}\right)\Bigg]^{-1}.\end{aligned}$$

In three summands we can factor out $N - D_n\left(\tau/N\right)$ and use the estimate $N - D_n\left(\tau/N\right) \ge N\tau^2$. Additionally, in the third summand we use the rough bound $N - D_n\left(\tau/N\right) \le N$ and in the fourth $\tau \le 1$. The same argument as in (3.6) shows that this also bounds the maximum in (4.3) and we get the result. ∎

Theorem 4.9 (Upper bound)**.** *Under the conditions of Definition 4.1 with $c = 1$, $\rho \ge \rho_{\min} = 25(\log(\lfloor \frac{M}{4} \rfloor) + 1)$, we have*

$$\operatorname{cond}(A) < \frac{5}{\tau}.$$

Proof. Direct inspection gives monotonicity of $C(\rho, M)$ with respect to ρ and also the estimate $C(25(\log(\lfloor M/4 \rfloor) + 1), M) \le C(25, 4) \le 12$. Hence $\left\|K^{-1}\right\| \le 12N^{-1}\tau^{-2}$ and together with the bound $\|K\| \le 52N/25$ from Lemma 4.3 we obtain the result. ∎

Remark 4.10. *Lower and upper bounds in Theorems 2.2 and 4.7 yield*

$$\frac{1}{\tau} \le \operatorname{cond}(A) \le \frac{5}{\tau}$$

for $\tau \le \frac{1}{4c^2}$ and $10c^2(\log(\lfloor M/4 \rfloor) + 1) \le \rho$. Some further comments:

i) The condition on ρ implies that at most $M = \mathcal{O}(N/\log N)$ nodes can be placed on the unit circle. Furthermore, let $\delta := \tau/N$ denote the minimal separation distance and $\Delta := \rho/N$ the minimal separation distance of non-colliding nodes, then the assumptions of Theorem 4.7 read as

$$\frac{10c^2(\log(\lfloor \frac{M}{4} \rfloor) + 1)}{\Delta} \leq N \leq \frac{1}{4c^2\delta},$$

which can be directly compared to [3, Cor. 1.1].

ii) Due to Lemma A.5, the upper bound from Theorem 4.7 remains valid if nodes are removed. Adding a column to a complex matrix with full row rank and more columns than rows increases $\sigma_{\min}$ and $\sigma_{\max}$ at most. Hence, if the number of columns N is even, A with one column added and one column deleted, respectively, can be used to establish bounds on the condition number.

iii) Finally note that the constant $\tilde{C}(\tau, \rho, c, M)$ in Lemma 4.6 being positive necessarily asks for $\tau \leq \frac{6\sqrt{2}}{c^2\pi^2} \approx \frac{0.85}{c^2}$ by letting $\rho \to \infty$.

5. Numerical Examples

All computations were carried out using MATLAB R2017b. As a test for the bounds in the case of one pair of nearly-colliding nodes we use the following configuration. Let the number of nodes $M = 20$ and $M = 200$ be fixed, respectively. Moreover, we choose $N = 1 + 12(M-1)$ which ensures that all nodes fit on the unit interval. We choose $\tau \in [10^{-11}, 1]$ logarithmically uniformly at random and $\rho_3, \ldots, \rho_M \in [6, 12]$ uniformly at random. Then we set the nodes $t_1 < \cdots < t_M \in [0, 1)$ such that $t_1 = 0$, $t_2 = \tau/N$ and for $j = 3, \ldots, M$, $|t_j - t_{j-1}| = \rho_j/N$. Afterwards the condition number of the corresponding Vandermonde matrix is computed. This procedure is repeated 100 times and the results are presented in Figure 5.1.

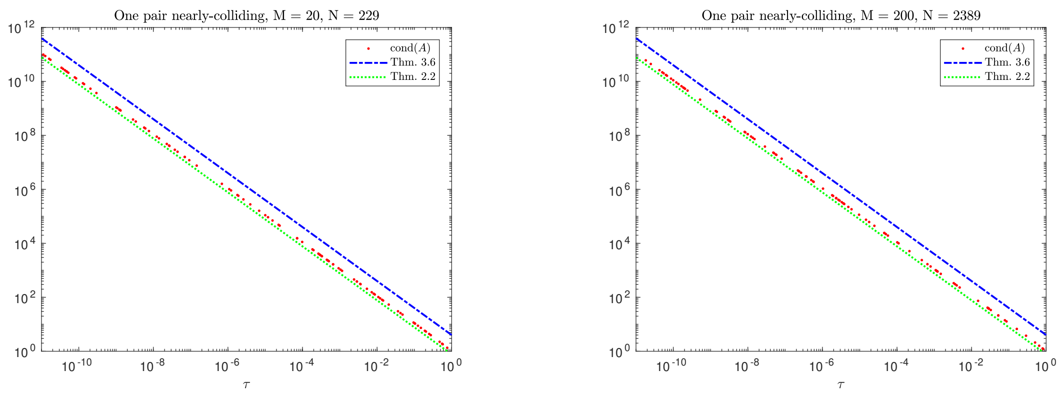


Figure 5.1: Numerical experiments for one pair of nearly-colliding nodes. Lower bound from Thm. 2.2, upper bound from Thm. 3.6.

For pairs of nearly-colliding nodes, we use the following configuration. Let the number of nodes $M = 20$ and $M = 200$ be fixed, respectively. Moreover, we choose the parameter $c = 2$ and τ_{max} and $\rho_{\min}$ as in Theorem 4.7. To ensure that all nodes fit on the unit interval, we choose N as the smallest odd integer bigger than $(c\tau_{max} + 2\rho_{\min})M/2$. Then we choose $\tau \in \left[10^{-11}, \tau_{max}\right]$ logarithmically uniformly at random and set the nodes $t_1 < \cdots < t_M \in [0,1)$ such that $t_1 = 0$, $t_2 = \tau/N$ and for $j = 3, \ldots, M$, $|t_j - t_{j-1}| = \rho_j/N$ if j is odd or $|t_j - t_{j-1}| = \tau_j/N$ if j is even, where $\tau_j \in [\tau, c\tau]$ and $\rho_j \in [\rho_{\min}, 2\rho_{\min}]$ are picked uniformly at random respectively. Afterwards the condition number of the corresponding Vandermonde matrix is computed. This procedure is repeated 100 times and the results are presented in Figure 5.2.

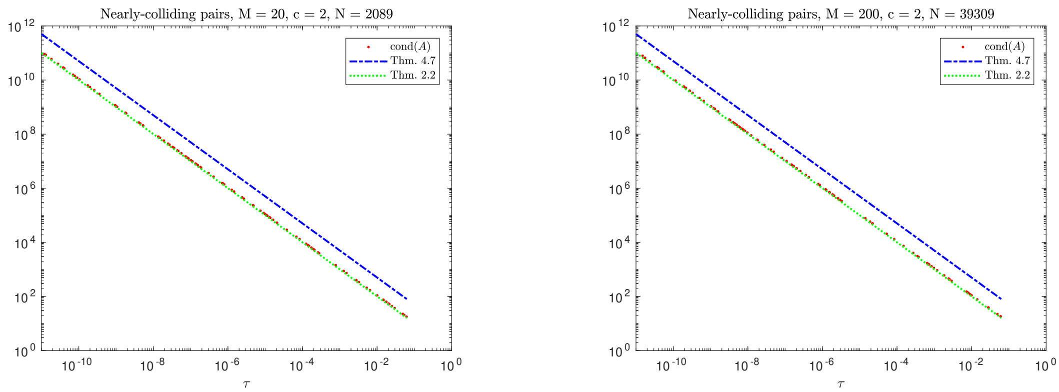


Figure 5.2: Numerical experiments for pairs of nearly-colliding nodes. Lower bound from Thm. 2.2, upper bound from Thm. 4.7.

In order to compare Theorem 4.7 with the results from [3, Cor. 1.1], we need to satisfy the assumptions of both results. We take $M = 3$ nodes with two nodes nearly-colliding, i.e. $t_1 = 0$, $t_1 = \tau/N$ and $t_2 = t_1 + \rho/N$. The assumptions in [3, Cor. 1.1] make it necessary that the nodes lie on an interval of length $\frac{1}{2M^2} = \frac{1}{18}$. We choose the parameter $c = 1$, $\rho_{\min} = 12$, $\tau_{max} = \frac{1}{4}$, and $N = 1001$. Then we pick $\tau \in \left[10^{-11}, \tau_{max}\right]$ logarithmically uniformly at random and $\rho \in \left[\rho_{\min}, \frac{N}{2M^2} - \tau\right]$ uniformly at random. Afterwards the inverse of the smallest singular value (norm of Moore-Penrose pseudo inverse) of the corresponding Vandermonde matrix is computed. This procedure is repeated 100 times and the results are presented in Figure 5.3(left).

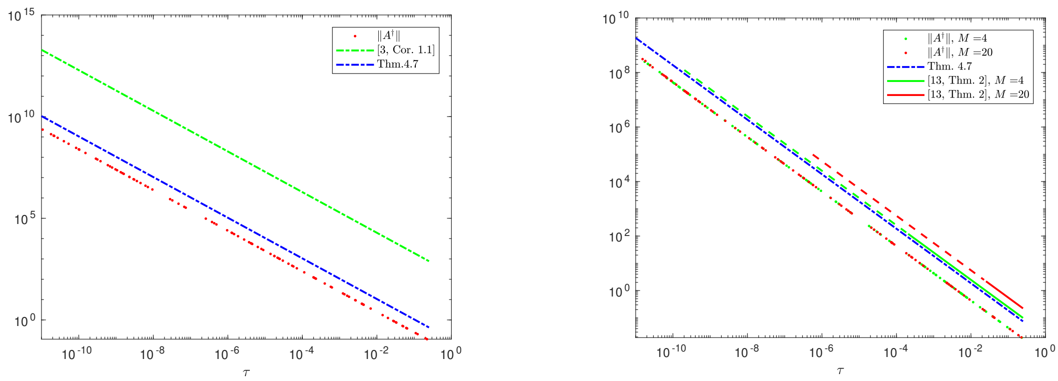


Figure 5.3: Upper bounds for $\left\|A^{\dagger}\right\|$; Left: Comparison of Thm. 4.7 with [3, Cor. 1.1]; Right: Comparison of Thm. 4.7 with [13, Thm. 2].

In order to compare our results with the ones from the second version of [13, Thm. 2], we set the parameter $N = 2^{15}+1$, $c = 1$, $\tau_{max} = \frac{1}{4}$ and $M = 4$ and $M = 20$, respectively. All pairs of nodes are placed uniformly, such that $t_j = \frac{2j-2}{M}$ and $t_{j+M/2} = t_j + \frac{\tau}{N}$ for $j = 1, \dots, \frac{M}{2}$, where τ is picked logarithmically uniformly at random from $[10^{-11}, \tau_{max}]$. Afterwards the inverse of the smallest singular value (norm of Moore-Penrose pseudo inverse) of the corresponding Vandermonde matrix is computed. This procedure is repeated 100 times and the results are presented in Figure 5.3(right). Note that in the second version of [13, Thm. 2, (2.5)] restrictions to τ are made. For pairs of nearly-colliding nodes

$$\tau \geq \frac{20^2 M 2^5 N^3}{\rho^2 (N-1)^3} \approx \begin{cases} 1.9 \cdot 10^{-4}, & M = 4, \\ 2.4 \cdot 10^{-2}, & M = 20, \end{cases} \tag{5.1}$$

is necessary, where we used the uniform bound $\rho < \frac{2N}{M}$ in our numerical example. In fact, the proof in the second version of [13, A.2] allows for improving the condition on τ, which is then

$$\tau \geq \frac{10^4 2^{10} M N^5}{\rho^4 \pi (N-1)^5} \approx \begin{cases} 1.8 \cdot 10^{-10}, & M = 4, \\ 5.6 \cdot 10^{-7}, & M = 20. \end{cases} \tag{5.2}$$

The results for τ satisfying (5.1) are shown in Figure 5.3(right) by proper lines and by broken lines for τ satisfying the improved condition (5.2).

Acknowledgements

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A. Appendix

The following technical results are used within the proofs of our main results.

Lemma A.1. *Let $n \in \mathbb{N}$, $N = 2n+1$, then the Dirichlet kernel* (2.2) *is bounded by*

$$N - \frac{\pi^2}{6} N^3 t^2 \leq D_n(t) \leq N - N^3 t^2, \qquad 0 \leq |t| \leq \frac{1}{N}.$$

Furthermore, the Dirichlet kernel and its first two derivatives are bounded by

$$|D_n(t)| \leq \frac{1}{2|t|},$$

$$\left|D_n'(t)\right| \leq N^2\left(\frac{\pi}{2N|t|}+\frac{1}{2N^2|t|^2}\right),$$
$$\left|D_n''(t)\right| \leq N^3\left(\frac{\pi^2}{2N|t|}+\frac{\pi}{N^2|t|^2}+\frac{1}{N^3|t|^3}\right)$$

for $0<|t|\leq 1/2$.

Proof. Due to symmetry it suffices to prove all bounds for $t>0$ and we use the explicit expression of the Dirichlet kernel in (2.2). The lower bound on $D_n(t)$ can be derived from the inequalities $x-x^3/6\leq\sin(x)\leq x$, that hold for all $x\in[0,\pi]$. The left inequality with $x=N\pi t$ and the right inequality with $x=\pi t$ lead to

$$\sin(N\pi t)\geq\left(N-\frac{\pi^2}{6}N^3t^2\right)\pi t\geq\left(N-\frac{\pi^2}{6}N^3t^2\right)\sin(\pi t).$$

The upper bound on $D_n(t)$ can be derived from the inequality $\cos(\alpha x)\leq\cos(x)$ that holds for all $x\in[0,\pi/2]$ and $\alpha>1$ such that $\alpha x\in[0,\pi/2]$. Integrating this inequality, choosing $\alpha=N/2$ and $x=\pi t$, and applying the double angle formula yields

$$\frac{\sin(N\pi t)}{2\cos(\frac{N}{2}\pi t)}=\sin\left(\frac{N}{2}\pi t\right)\leq\frac{N}{2}\sin(\pi t).$$

Reordering the inequality and applying that $\cos(x)\leq 1-4x^2/\pi^2$ for all $x\in[0,\pi/2]$, yields

$$\frac{\sin(N\pi t)}{\sin(\pi t)}\leq N\cos(\frac{N}{2}\pi t)\leq N(1-N^2t^2).$$

Finally, the remaining bounds on the absolute values can be proven by calculating the first and second derivatives and using $\sin(x)\geq 2x/\pi$ and $\cot x\leq 1/x$ that hold for all $x\in(0,\pi/2]$. ■

Lemma A.2. *Let* $M,\widetilde{M}\in\mathbb{C}^{m\times n}$ *with* $|M|_{j,\ell}\leq\widetilde{M}_{j,\ell}$ *for all* $j=1,\ldots,m$, $\ell=1,\ldots,n$, *then*

$$\|M\|\leq\left\|\widetilde{M}\right\|.$$

Proof. We directly show the result by

$$\|M\|=\max_{\|x\|=1}\|Mx\|=\max_{\|x\|=1}\sum_{k=1}^m\left|\sum_{\ell=1}^n M_{k\ell}x_\ell\right|^2\leq\max_{\|x\|=1}\sum_{k=1}^m\left(\sum_{\ell=1}^n|M_{k\ell}|\,|x_\ell|\right)^2$$
$$\leq\max_{\|x\|=1}\sum_{k=1}^m\left(\sum_{\ell=1}^n\widetilde{M}_{k\ell}\,|x_\ell|\right)^2=\max_{\|x\|=1}\sum_{k=1}^m\left(\sum_{\ell=1}^n\widetilde{M}_{k\ell}x_\ell\right)^2=\left\|\widetilde{M}\right\|.$$

Note that similar estimates can be found for the Frobenius norm in [9, p. 520]. ■

Lemma A.3 (Neumann expansion)**.** *Let* $M\in\mathbb{C}^{n\times n}$ *hermitian and positive definite. Let* $\eta\in\mathbb{R}$ *be a parameter satisfying* $\eta>\|M\|$, *then*

$$\|M^{-1}\|\leq\frac{1}{\eta-\|\eta I-M\|}.$$

Proof. Applying the Neumann series to the matrix $I - \eta^{-1}M$ yields

$$\left\|M^{-1}\right\| = \frac{1}{\eta}\left\|\sum_{k=0}^{\infty}\left(I - \frac{1}{\eta}M\right)^k\right\| \leq \frac{1}{\eta}\frac{1}{1 - \left\|I - \frac{1}{\eta}M\right\|} = \frac{1}{\eta - \|\eta I - M\|}.$$

■

Lemma A.4 (Schur complement decomposition, cf. [9, Eq. (0.8.5.3)])**.** *Let $n_1, n_2 \in \mathbb{N}$ and $M \in \mathbb{C}^{(n_1+n_2)\times(n_1+n_2)}$ be a 2×2 block matrix of the form*

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}, M_1 \in \mathbb{C}^{n_1\times n_1}, M_4 \in \mathbb{C}^{n_2\times n_2},$$

with M_1 being invertible. Then the Schur complement decomposition is given by

$$\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \begin{pmatrix} I_{n_1} & 0 \\ -M_3M_1^{-1} & I_{n_2} \end{pmatrix}^{-1}\begin{pmatrix} M_1 & 0 \\ 0 & M_4 - M_3M_1^{-1}M_2 \end{pmatrix}\begin{pmatrix} I_{n_1} & -M_1^{-1}M_2 \\ 0 & I_{n_2} \end{pmatrix}^{-1}.$$

The block $[M/M_1] := M_4 - M_3M_1^{-1}M_2$ is called Schur complement of M_1 in M.

Lemma A.5 (Cauchy interlacing theorem for eigenvalues (inclusion principle) cf. [9, Thm. (4.3.28)])**.** *Let $M \in \mathbb{C}^{n\times n}$ be a Hermitian complex matrix, such that*

$$M = \begin{pmatrix} M_1 & M_2 \\ M_2^* & M_3 \end{pmatrix}, \quad M_1 \in \mathbb{C}^{m\times m}, M_2 \in \mathbb{C}^{m\times(n-m)}, M_3 \in \mathbb{C}^{(n-m)\times(n-m)}.$$

Let the eigenvalues of M and M_1 be ordered in non-decreasing order, then

$$\lambda_i(M) \leq \lambda_i(M_1) \leq \lambda_{i+n-m}(M), \quad i = 1, \ldots, m.$$

Lemma A.6 (Block Gerschgorin theorem, cf. [9, 6.1.P17] or [8, Thm. 5])**.** *Let $M \in \mathbb{C}^{nm\times nm}$ be an $m\times m$ block matrix with blocks $M_{ik} \in \mathbb{C}^{n\times n}$. Let the diagonal blocks M_{ii} be normal and denote $\lambda_1^{(i)}, \ldots, \lambda_n^{(i)}$ their eigenvalues respectively. Then the eigenvalues of M are included in the set*

$$\bigcup_{i=1}^{n}\bigcup_{j=1}^{m}\{z \in \mathbb{C}\colon |z - \lambda_j^{(i)}| \leq \sum_{k\neq i}\|M_{ik}\|\}.$$

In particular, we have for $M \in \mathbb{C}^{m\times n}$ the inequalities

$$\left\|\begin{pmatrix} 0 & M^* \\ M & 0 \end{pmatrix}\right\| \leq \|M\|, \text{ and } \qquad \left\|\begin{pmatrix} I & 0 \\ M & I \end{pmatrix}\right\|^2 \leq 1 + \|M\| + \|M\|^2.$$