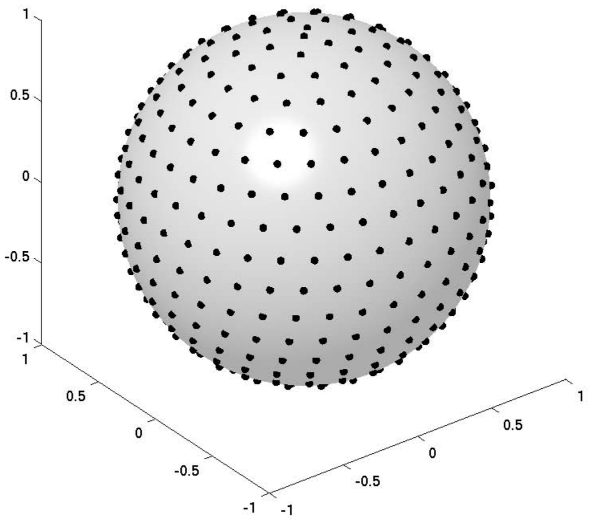
Efficient reconstruction of functions on the sphere from scattered data

Jens Keiner[1], **Stefan Kunis**[2], and **Daniel Potts**[2,*]

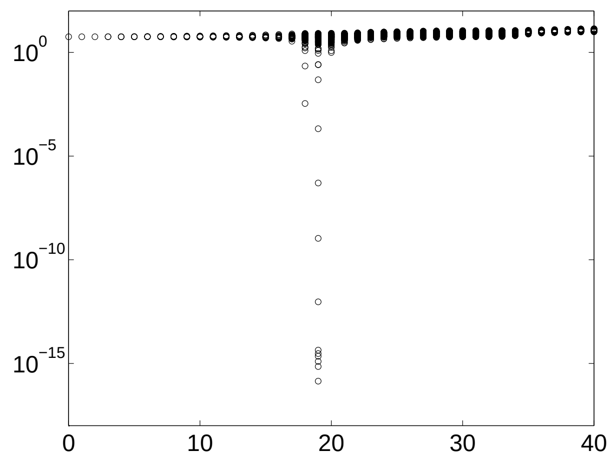
[1] Universität zu Lübeck,Institut für Mathematik, Wallstrasse 40, 23560 Lübeck, Germany
[2] Technische Universität Chemnitz, Fakultät für Mathematik, Reichenhainer Str. 39, 09107 Chemnitz, Germany

Motivated by the fact that most data collected over the surface of the earth is available at scattered nodes only, the least squares approximation and interpolation of such data has attracted much attention, see e.g. [1, 2, 5]. The most prominent approaches rely on so-called zonal basis function methods [16] or on finite expansions into spherical harmonics [12, 14]. We focus on the latter, i.e., the use of spherical polynomials since these allow for the application of the fast spherical Fourier transform, see for example [8, 9] and the references therein.

If we consider the problem of reconstructing a spherical polynomial of degree $N \in \mathbf{Z}$ from sample values, one might set up a linear system of equations with $M = (N+1)^2$ interpolation constraints which has to be solved for the unknown vector of Fourier coefficients $\hat{\boldsymbol{f}} \in \mathbb{C}^{(N+1)^2}$. If the nodes for interpolation are chosen such that the interpolation problem has always a unique solution, the sampling set is called a fundamental system. As can be seen in Figure 1(b), also geometrically well distributed nodes on the sphere can lead to an ill conditioned square spherical Fourier matrix. Hence, we relax the condition that the number of equations M coincides with the number of unknowns $(N+1)^2$. Considering the overdetermined case $M > (N+1)^2$ or the underdetermined case $M < (N+1)^2$ leads to far better distributed singular values of the system matrix as seen in Figure 1(b).



(a) Generalised spiral nodes [15].



(b) Singular value distribution.

Fig. 1 Distribution of the singular values of the spherical Fourier matrix $\boldsymbol{Y} \in \mathbb{C}^{M\times(N+1)^2}$ with respect to the polynomial degrees $N = 0, \ldots, 40$ for $M = 400$ generalised spiral nodes, cf. [7].

Our main result [7] is that for given sampling nodes the polynomial degree N can either be chosen small enough with respect to the inverse mesh norm or large enough with respect to the inverse separation distance of the sampling set to ensure a well conditioned spherical Fourier matrix. In both cases, the derived conditions are optimal up to a moderate constant.

In the first part, we consider the overdetermined case $M > (N+1)^2$, that is the least squares approximation by spherical polynomials to given data. The approximation by (univariate) trigonometric polynomials has been proven to be stable if the polynomial degree is less than the inverse of the mesh norm of the sampling set in [6]. Subsequently, Feichtinger, Gröchenig and Strohmer [3] developed the celebrated adaptive weights, conjugate gradient, Toeplitz method (ACT) for the fast iterative solution of the least squares problem. Moreover, so-called spherical L^p-Marcinkiewicz-Zygmund inequalities due to Mhaskar,

* Corresponding author E-mail: potts@mathematik.tu-chemnitz.de

Narcowich, Ward [11] and Filbir, Themistoclakis [4] yield stable least squares approximation for a quasi-uniform subset of a dense sampling set. We generalise the idea of adaptive weights to the sphere and have the following L^2-Marcinkiewicz-Zygmund inequality for dense sampling sets.

Theorem 0.1 *Let a sampling set $\mathcal{X} \subset \mathbb{S}^2$ of cardinality $M \in \mathbb{N}$ with mesh norm $\delta = 2 \max_{\boldsymbol{\xi} \in \mathbb{S}^2} \min_{j=0,\dots,M-1} \arccos(\boldsymbol{\xi}_j, \boldsymbol{\xi})$ be given. Moreover, let the polynomial degree $N \in \mathbb{N}$ be bounded by $154N\delta < 1$ and set up so called Voronoi weights $w_j > 0$. Then we have for arbitrary spherical polynomials $f = \sum_{k=0}^{N} \sum_{n=-k}^{k} \hat{f}_k^n Y_k^n(\boldsymbol{\xi})$ the weighted norm estimate*

$$(1 - 154N\delta) \|f\|_{L^2(\mathbb{S}^2)}^2 \leq \sum_{j=0}^{M-1} w_j |f(\boldsymbol{\xi}_j)|^2 \leq (1 + 154N\delta) \|f\|_{L^2(\mathbb{S}^2)}^2 .$$

Under the above conditions and for given data $\boldsymbol{y} \in \mathbb{C}^M$, the least squares problem

$$\min_f \sum_{j=0}^{M-1} w_j |y_j - f(\boldsymbol{\xi}_j)|^2$$

has condition number $\kappa \leq (1 + 154N\delta)/(1 - 154N\delta)$.

On the other hand, we focus on the underdetermined case $M < (N+1)^2$, where we aim to interpolate the given data. The trigonometric interpolation problem has been considered in [10], from where we adopt a smoothness-decay principle to construct strongly localised zonal polynomials with strictly positive Fourier-Legendre coefficients. In conjunction with a refined version of the packing argument from [13], we prove stable interpolation for well separated sampling sets in the following.

Theorem 0.2 *Let a sampling set $\mathcal{X} \subset \mathbb{S}^2$ of cardinality $M \in \mathbb{N}$ with separation distance $q = \min_{0 \leq j < l < M} \arccos(\boldsymbol{\xi}_j, \boldsymbol{\xi}_l)$ be given. Moreover, let the polynomial degree $N \in \mathbb{N}$ be bounded by $Nq > 14$ and set up so called B-spline weights $\hat{w}_k > 0$. Then the matrix $\boldsymbol{K} = (K_{j,l})_{j,l=0,\dots,M-1}$, $K_{j,l} = \sum_{k=0}^{N} \sum_{n=-k}^{k} \hat{w}_k Y_k^n(\boldsymbol{\xi}_j) \overline{Y_k^n(\boldsymbol{\xi}_j)}$ has bounded eigenvalues*

$$|\lambda(\boldsymbol{K}) - 1| \leq \left(\frac{14}{Nq} \right)^3 .$$

Under the above conditions and for given data $\boldsymbol{y} \in \mathbb{C}^M$, the optimal interpolation problem

$$\min_f \sum_{k=0}^{N} \sum_{n=-k}^{k} \frac{\left| \hat{f}_k^n \right|^2}{\hat{w}_k} \quad \text{subject to} \quad \sum_{k=0}^{N} \sum_{n=-k}^{k} \hat{f}_k^n Y_k^n \left(\boldsymbol{\xi}_j \right) = y_j$$

has condition number $\kappa \leq (1 + (14/Nq)^3)/(1 - (14/Nq)^3)$.

References

[1] J. P. Boyd. *Chebyshev and Fourier Spectral Methods*. Dover Press, New York, second edition, 2000.

[2] G. E. Fasshauer and L. L. Schumaker. Scattered data fitting on the sphere. In M. Dahlen, T. Lyche, and L. L. Schumaker, editors, *Mathematical Methods for Curves and Surfaces II*, pages 117 – 166, Nashville, 1998. Vanderbilt University Press.

[3] H. G. Feichtinger, K. Gröchenig, and T. Strohmer. Efficient numerical methods in non-uniform sampling theory. *Numer. Math.*, 69:423 – 440, 1995.

[4] F. Filbir and W. Themistoclakis. Polynomial approximation on the sphere using scattered data. Preprint 06-12, GSF München, submitted, 2006.

[5] W. Freeden, T. Gervens, and M. Schreiner. *Constructive Approximation on the Sphere*. Oxford University Press, Oxford, 1998.

[6] K. Gröchenig. Reconstruction algorithms in irregular sampling. *Math. Comput.*, 59:181 – 194, 1992.

[7] J. Keiner, S. Kunis, and D. Potts. Efficient reconstruction of functions on the sphere from scattered data. *J. Fourier Anal. Appl.*, 13:435 – 458, 2007.

[8] J. Keiner and D. Potts. Fast evaluation of quadrature formulae on the sphere. *Math. Comput.*, 2007. accepted.

[9] S. Kunis and D. Potts. Fast spherical Fourier algorithms. *J. Comput. Appl. Math.*, 161:75 – 98, 2003.

[10] S. Kunis and D. Potts. Stability results for scattered data interpolation by trigonometric polynomials. SIAM *J. Sci. Comput.*, 29:1403 – 1419, 2007.

[11] H. N. Mhaskar, F. J. Narcowich, and J. D. Ward. Spherical Marcinkiewicz-Zygmund inequalities and positive quadrature. *Math. Comput.*, 70:1113 – 1130, 2001. Corrigendum on the positivity of the quadrature weights in 71:453 – 454, 2002.

[12] C. Müller. *Spherical Harmonics*. Springer, Aachen, 1966.

[13] F. J. Narcowich, N. Sivakumar, and J. D. Ward. Stability results for scattered-data interpolation on euclidean spheres. *Adv. Comput. Math.*, 8:137 – 163, 1998.

[14] M. Reimer. *Multivariate Polynomial Approximation*. Birkhäuser Verlag, Basel, 2003.

[15] E. B. Saff and A. B. J. Kuijlaars. Distributing many points on a sphere. *Math. Intelligencer*, 19:5 – 11, 1997.

[16] H. Wendland. *Scattered Data Approximation*. Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press, Cambridge, 2005.